ELECTRIC FILTERS

A SIMPLIFIED TREATMENT OF THE FOUR-TERMINAL NETWORKS COMMONLY USED IN TELEPHONE WORK, COVERING FILTERS, ATTENUATORS, PHASE SHIFTING NETWORKS, AND ATTENUATION EQUALIZERS

BY

T. H. TURNEY, Ph.D.
RESEARCH DEPARTMENT, AUTOMATIC TELEPHONE AND ELECTRIC CO , LTD.

LONDON
SIR ISAAC PITMAN & SONS, LTD.
1947

First Edition, 1946
Reprinted, 1947

SIR ISAAC PITMAN & SONS, LTD.
PITMAN HOUSE, PARKER STREET, KINGSWAY, LONDON, W C 2
THE PITMAN PRESS, BATH
PITMAN HOUSE, LITTLE COLLINS STREET, MELBOURNE
UNITEERS BUILDING, RIVER VALLEY ROAD, SINGAPORE
27 BECKETTS BUILDINGS, PRESIDENT STREET, JOHANNESBURG

ASSOCIATED COMPANIES
PITMAN PUBLISHING CORPORATION
2 WEST 45TH STREET, NEW YORK
205 WEST MONROE STREET, CHICAGO
SIR ISAAC PITMAN & SONS (CANADA), LTD
(INCORPORATING THE COMMERCIAL TEXT BOOK COMPANY)
PITMAN HOUSE, 381-383 CHURCH STREET, TORONTO

BOOK PRODUCTION WAR ECONOMY STANDARD

THIS BOOK IS PRODUCED IN
COMPLETE CONFORMITY WITH THE
AUTHORIZED ECONOMY STANDARDS

MADE IN GREAT BRITAIN AT THE PITMAN PRESS, BATH
D7—(T.271)

PREFACE

Filters have begun to arouse a great deal of interest, and it is thought that there is room for a book treating the subject in as simple a manner as possible, i.e. in an explanatory way.

The writer has aimed at giving an introduction to each branch of the subject, including the Cauer method of design.

The various circuits in use, such as, say, band pass filter sections, are not all separately worked out. This has been done by others quite fully. Only so much has been included as to show the method of handling the subject; and the book is intended more as a textbook than as a reference book. Chapter X on Cauer filters must be regarded as a stopgap. The Conclusion is the author's view of the way future development ought to be attempted, indicating the lines to go upon.

The author believes he has made a discovery in this subject. It is that if you go out and buy any number of any sizes of coils and condensers, bring them home and connect them up in any way —put two terminals anywhere in the circuit for input and two more terminals anywhere else for output—you have built a filter. There may be very many pass bands at various frequencies, and the filter may have curious characteristic impedances in each band. Yet it is a true filter, and gives the quite remarkable results common to all filters. Tested at any frequency in a pass band, it will (apart from losses in coils and condensers) give no weakening of voltage and current, no attenuation, that is, and this is true, however many similar circuits you send current through all put in a row.

If tested at any frequency in an attenuation band, however, it will have some attenuation; and it will only be necessary to build sufficient similar circuits and put them end to end to get any desired attenuation. There are details to be observed in understanding this general theorem, such as, for example, that to experience no attenuation one ought to make the load resistance of the correct value. Yet the statement stands, in its surprising breadth and generality.

In order to get the pass bands where one desires as regards frequency, and in order to suit a given load resistance, one needs to understand design. It is the purpose of this book to teach it. Of what use then is the author's generalization? This much: it is not an accident that the circuits described by their inventors as

filters are filters; they must be so. They cannot help but be; and since all circuits built of reactances are filters, it is plain that there is an infinite field for research and design.

The author desires to thank Mr. Wm. Emery of the Liverpool City Technical College for help in designing the illustrations, the finished drawings for which were prepared by Mr. B. C. Wood.

He is greatly indebted to Mr. J. B. McCusker of the A.T. & E. Co. for reading the proofs, and also to Mr. C. Rhodes, M.Eng., A.M.I.E.E., and to the Publishers. Their work has helped to keep the book free from the sort of mistake that students find perplexing.

INTRODUCTION

MANY pieces of electrical apparatus, such as lamps, bells, radiators, motors and resistances, as well as coils and condensers, have two terminals, while a transformer has four.

In telephone and telegraph work many pieces of apparatus are found which have four terminals like a transformer, so they take a certain current at a certain voltage and give out a certain current at perhaps a different voltage. An attenuator composed of pure resistances is a case in point. It has two input and two output terminals.

Attenuators composed of pure resistances are now much used in transmission testing. They are simple to make and they produce simple and definite effects because they attenuate all frequencies alike.

In the early days of telephone work, transmission testing was done with artificial cables composed of resistances and condensers to simulate the resistance and capacity of cables with their variable attenuation.

When loading became general, however, most cables had a fairly level attenuation frequency curve, and an artificial line or attenuator was needed which also had a level characteristic.

The solution to this was to build attenuators with pure resistances.

A filter has also four terminals and enables some frequencies to pass freely while being a barrier to others. Other networks merely shift the phases of current.

If a circuit contains resistances like an attenuator, and also condensers and coils like a filter, it may have an effect something between a filter and an attenuator; it may attenuate different frequencies in a different degree. It is then called an attenuation equalizer, and is useful for putting in a transmission line to level up the variation in attenuation of the line with frequency.

To revert to filters, the filter in its pass band shifts the phases of the currents flowing through it. If the filter has a pass band from 0 to infinity it is then called a phase shift network, as it does not attenuate any frequency, but only shifts the phases of currents passing through it. Such networks can undo the harmful effects of phase shifts in transmission lines. In general, a line 1000 miles long tends to produce transients due to phase shift, which only begin to be

harmful for speech at that length. On the other hand, picture transmission is much upset by phase shift in the cable, and phase shifting networks can restore sharpness to the picture if suitably designed.

All these networks are distinguished by having four terminals, and their study may be called that of four-terminal networks.

The Fundamental Plan of the Work

These networks are introduced into the line to cause certain effects, such as a filtering action, and with filters one has to take into account the reflections at the ends of the filter, as shown in Chapter IV. This, however, is difficult, so one first ignores them by working on chains of similar sections infinitely long, i.e. one studies *characteristic impedances* and *propagation constants*: these are the "bricks" to work with.

A further simplification is to ignore, first of all, the resistance losses in coils and condensers, which are treated in Chapter IV. This means that one has to study circuits built up of reactances only.

The circuits must have some shape when made into a drawing, i.e. some geometrical pattern, so one studies circuit forms such as ladders, lattices, and gates. It is simpler to use resistances instead of reactances in deriving preliminary formulas for the *characteristic impedances* and for the *propagation constant*. Here, then, is the beginning. Using resistances first simplifies the algebra, and also makes up a very practical piece of apparatus—the *attenuator*, dealt with in Chapter I.

Then reactances can be used to make filters, as in Chapters II and III. Later, losses in coils can be taken into account by using complex quantities as in Chapter IV. After that, the study of reflections shows what the circuits will do when connected up to other apparatus in the usual practice, as shown in Chapter IV.

This is the work involved. Other networks are dealt with, and also methods of measurement.

Towards the end there is a chapter on fundamental theorems. Following Euclid, this ought to be at the beginning; but one understands general theorems easily by observing particular cases first. As a concession to the reader, who must be the judge, this chapter is, therefore, not put at the beginning, where it logically belongs.

Some account of Cauer's work is also given, using Tschebbytcheff polynomials.

CONTENTS

PAGE

PREFACE iii

INTRODUCTION v

TABLES ix

CHAPTER I

ATTENUATORS 1

Geometrical progression and the exponential—The resistance values for a "T" attenuator—Resistance coil winding—Winding low resistances—Networks balanced to earth

CHAPTER II

LADDER FILTERS. 12

Double-shunt terminated ladder—Formulas for components of simple ladder low pass filter—Impedance of low pass filter—Half-series case—Double-shunt case—The attenuation of the low pass filter—Formula for the attenuation of a ladder filter—Proof of the formula for the propagation constant of ladder filters—An easy proof to the impedance of a length of line open and closed at the far end—The use of the attenuation formula—A simple explanation of the action of a low pass filter—Phase changes in filters—The attenuation curve of the low pass ladder filter—The use of "general" curves—The relation between cosh *P* and attenuation—The high pass filter—The two terminations for ladder high pass filters—High pass filter balanced to earth—The impedance of the high pass filter—How to tell the impedance curve from the circuit—Ladder band pass filter—The components of the band pass filter—Terminations of ladder band pass filters—Simple explanation of this band pass filter in action—Balance to earth of band pass filters—Derived ladder filters—The derived low pass ladder filter—Deriving high and band pass filters—Shunt-derived filters—The method of derivation and termination as it affects impedance in the pass band—Impedances of filters in the attenuation band—Reciprocal impedances—Building up filters from sections

CHAPTER III

LATTICE FILTERS 66

Equivalent lattice and ladder formulas—The criterion for a pass band in lattice filters

CHAPTER IV

LOSSES IN COMPONENTS 73

Effect of losses in filters—Dealing with vectors nearly 0° or 90° or 180°—The use of charts—Circuit components—General remarks on losses—"Insertion losses"—The case of differing ends—The use of the chart (Fig. 50)—Theorem on the effect of terminal reflections

CHAPTER V

PAGE

THE EQUIVALENCE OF NETWORKS 88

Ladder and lattice networks—The lattice and the gate network—The gate and the ladder—Bridged "T" networks—The star mesh theorem—The use of mutual inductance in filter design—Mutual inductance in lattices—The tuned transformer band pass filter

CHAPTER VI

PHASE SHIFT NETWORKS 100

The use of phase shifting networks—Design of phase networks—Simple lattice phase shift network

CHAPTER VII

ATTENUATION EQUALIZERS 107

Desirable qualities for an attenuation equalizer—The constant resistance (one-way) "L" equalizer—Proof of constancy of resistance for constant resistance "L" equalizer—The voltage reduction ratio of this circuit—Design of arm—Simple example of the "L" constant resistance (one-way) equalizer—The construction of the inverse arm—Constant resistance (both ways) attenuation equalizers—A chart for the "L" equalizer—The construction of the chart

CHAPTER VIII

MEASUREMENTS 114

Bridges and measurements—A condenser as standard—The detector—Impedance relation in bridges—Proof to the negative series bridge—Negative shunt bridge—Positive shunt bridge—Positive series bridge—Robinson frequency bridge—Setting up a capacity standard—The hybrid coil bridge—The measurement of phases in four-terminal networks—Note on the measurement of attenuation of filters

CHAPTER IX

GENERAL THEOREMS 126

I: Thévénin's theorem—II: Maximum volt-ampere theorem—III: Pure resistance networks with four terminals—Corollary to Theorem III—IV: The filter theorem—V: Campbell's theorem—Corollary to Theorem V—VI: The impedance multiplier theorem—VII: Foster's theorem for reactances—VIII: Impedance frequency theorem—IX: The "change of tempo" theorem—X: Resistance in reactive networks—XI: Impedance inversion—XII: Frequency inversion—XIII: Networks with no attenuation—XIV: Generalized Kirchhoff's law—XV: Resistance results generalized—XVI: The reciprocal theorem—XVII: The linearity theorem—XVIII: Simplicity of alternating currents

CHAPTER X

CAUER FILTERS 139

The use of trigonometry for avoiding arithmetic—Impedance characteristics—The fundamental functions—Design of band pass filters

CHAPTER XI

PAGE
FILTERS USING CIRCUITS WITH UNUSUAL ELEMENTS . . . 158
Quartz crystal filters—Co-axial cable filters—Wave guide filters

CHAPTER XII

CONCLUSION 166

APPENDICES

I. DESIGN OF FILTERS FOR OPERATION OVER A RESTRICTED FREQUENCY RANGE 167

II. THE EFFECT ON THE PERFORMANCE OF A FILTER OF SLIGHT LACK OF ACCURACY IN COILS AND CONDENSERS . . . 168

INDEX 169

TABLES

PAGE
1. Relation between Voltage (and Current) Ratio and Decibels . . 2
2. The Values of the Resistances in the "T" Attenuator . . . 7
3. Impedance of Low Pass Simple Ladder Filter above Cut-off (Half-series Terminated) 19
4. Rising Impedance in the Pass Band (Low Pass Filter Double-shunt Ended) 20
5. Impedance in the Attenuation Band (Low Pass Filter Double-shunt Ended) 20
6. Attenuation Curve of the Simple Low Pass Ladder Filter and the Phase Change in the Pass Band 33
7. Components for (Nominal) 600-ohm Impedance, High Pass Filters . 39
8. Series Condenser and Shunt Coil for a Band Pass Filter 100 Cycles Wide, having 600 ohms Impedance 47
9. Table for Calculating Values of Inductance and Capacity of Ladder Filters 49
10. Impedance of Filters in the Pass Band 54
11. Impedance of Filters (Ladder-derived) in the Attenuation Band . 55
12. Series-derived Low Pass Ladder Filters 55
13. Impedances of Filters in the Pass Band 62
14. General Formulas 71
15. A Useful Function 80
16. Simple Lattice Phase Shift Network 102
17. More Complicated Phase Network 105

		PAGE
18.	Attenuation Equalizer	111
19.	Cauer's Class I' Filter	143
20.	Table of Functions for Easy Arithmetic	149
21.	Single Section of Derived Ladder Filter	154
22.	Properties of Q (1). Data for a Class I Filter	155
23.	Co-axial Cable Filters	163
24.	The Analogy between Wave Guide Resonators and Ordinary Tuned Circuits	164

ELECTRIC FILTERS

CHAPTER I

ATTENUATORS

ATTENUATORS made with pure resistances have the advantage that they give an attenuation* which does not depend on frequency. Also the characteristic impedance is a constant resistance, and so is the impedance looking into the attenuator when the far end is closed with the proper resistance, whatever the frequency and whatever the attenuation, if the attenuator is built correctly. Thus voltage calculations become easy.

The resistances are calculated from the voltage and current attenuation ratio, which may be called N. This is the ratio with the load connected; and the value of N corresponding to so many decibels is given by the formula

$$\text{db} = 20 \log_{10} N$$

$$\text{or } N = \text{antilog}_{10}\left(\frac{\text{db}}{20}\right)$$

Note particularly that **transmission measurements are logarithms of current, voltage, and power ratios.**

A voltage or current ratio of 10 : 1 is 20 db, but a power ratio of 100 : 1 is also 20 db because power is voltage or current squared, provided the same load resistance is used. There are not two sorts of decibels, though one may write

$$\text{db} = 10 \log_{10}\left(\frac{W_1}{W_2}\right),$$

the W being watts. A table relating db to current and voltage ratio N is given on page 2, (see Table 1).

The desirable features of any attenuator are that the network must match a line of, say, R ohms. It must therefore give its attenuation when closed at the far end by R ohms, and under this condition the impedance at the sending end is R ohms too. Attenuators may be fixed or variable.

* Attenuation simply means "weakening" of current.

TABLE 1

RELATION BETWEEN VOLTAGE (AND CURRENT) RATIO AND DECIBELS

N	db	db	N
0	Infinity	0	1
1	0	1	1·18
2	6·02	2	1·256
3	9·56	3	1.416
4	12·04	4	1.586
5	13·98	5	1.777
6	15·58	6	2
7	16·9	7	2.24
8	18·06	8	2.51
9	19·12	9	2.82
10	20	10	3.16
15	23·54	15	5.63
20	26·02	20	10
30	29·56	25	17.77
40	30·04	30	31.6
50	33·98	35	56.3
100	40	40	100
1,000	60	45	177.7
10,000	80	50	316
100,000	100	60	1,000
1,000,000	120	70	3,160
		80	10,000
		90	31,000
		100	100,000

The Π network in Fig. 1 has the properties that it reduces the current and voltage step by step along the network and, if it is

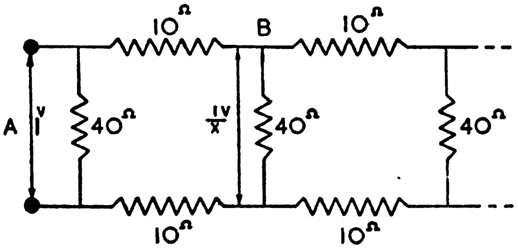


FIG. 1. INFINITE LADDER NETWORK

infinitely long, tends to have a certain definite "resistance" as measured at the two end terminals. That is to say, it has a "characteristic impedance," just like a cable.

That it also causes attenuation is instantly seen by the fact that

it is a repeated potentiometer. If it is long enough, there cannot be anything but a negligible current flowing at the far end, and so it does not matter what is connected to the far end; resistance measurements at the near end depend solely on the resistances in the sides and rungs of the ladder. In other words, there is a characteristic impedance which is *characteristic* of that network.

To prove that the reduction of current per section is the same definite factor all along the infinite network, consider the network in Fig. 1 to be infinitely long, then add another section at the beginning as in Fig. 2.

If 1 volt is applied to A in Fig. 1 the voltage at B may be $\frac{1}{x}$ volts, say. In Fig. 2, if a higher voltage x is applied to the beginning S there will be 1 volt at A because the two networks Fig. 1 and Fig. 2,

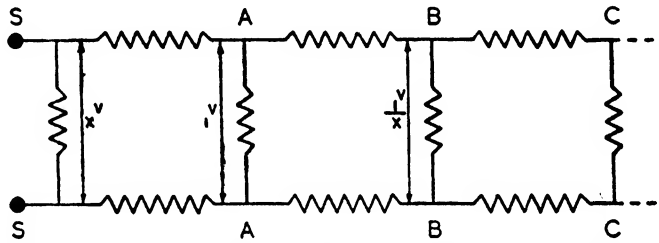


FIG. 2. CURRENTS IN LADDER NETWORK UNCHANGED BY ADDING A SECTION AND INCREASING THE SENDING END VOLTAGE

being supposed to be infinitely long, look alike, and because of the proportionality due to Ohm's law as expressed in the linearity theorem. (See Chapter IX, "General Theorems.")

Then the 1 volt at A in Fig. 2 will cause $\frac{1}{x}$ volts at B in the same figure because 1 volt at A caused $\frac{1}{x}$ volts at B in Fig. 1, and the A's and B's in the two figures correspond.

Finally, it is seen that the three voltages at the beginnings of the first three sections of the ladder, namely S and A and B in Fig. 2, are in the ratio $x : 1 : \frac{1}{x}$, which is a geometric progression. An extension of reasoning proves geometrical progression to hold throughout the network. The whole argument is that x volts at S must produce 1 volt at A in Fig. 2 because 1 volt at A in Fig. 1 produces $\frac{1}{x}$ volts at B in Fig. 1.

Geometrical Progression and the Exponential

The following is a little-known theorem on the exponential—

"Where there is geometrical progression, there is exponential decay, and logarithms are indicated." Hence a ladder network has a "propagation constant." The next step is to call the shunt resistances *b*, say, and half the series resistances *a*, say, and find one formula for characteristic impedance and another for propagation constant in terms of *a* and *b*. These formulas will then be proper to ladder networks.

If the network is closed with a resistance equal to its characteristic impedance, *R*, the ratio of voltage to current at the sending end is $\frac{E_s}{i_s} = R$ just as at the far end across the resistance, *R*, voltage divided by i_r gives *R* because of Ohm's law.

In other words,

$$\frac{E_s}{i_s} = \frac{E_r}{i_r}$$

which may be written

$$\frac{E_s}{E_r} = \frac{i_s}{i_r}$$

Here we have proved what was stated above that *N* is *either* the voltage *or* the current ratio; for they are equal to each other.

Finding the voltage ratio $\frac{E_s}{E_r}$ and equating it to *N* gives one of the equations or conditions for working out the resistances of the arms. The other condition is that the current ratio must also be *N*, and that follows if the characteristic impedance of the Π is equal to *R*, the resistance with which it is closed. Since, through the idea of characteristic impedance, the voltage ratio equals the current ratio, these give two equations, each with a ratio, which are equal to each other.

To work out the "T" attenuator, Fig. 3, let the series resistances be *a* ohms each and the shunt resistance *b* ohms; call the current in the resistance *R* at the end i_r. The voltage E_b across the shunt resistance is then $E_b = (R + a)i_r$. The resistance *R* is the resistance that the attenuator is working into, i.e. that for which it is designed. The current in *b*, which may be called i_b, is E_b divided by *b*

or

$$\frac{(R + a)i_r}{b}$$

The sent current is this, with i_r added, which gives

$$\frac{(R+a)}{b}i_r + i_r.$$

The sent current divided by i_r, the received current, is the current ratio, which may be called N.

Then

$$N = \frac{R+a}{b} + 1.$$

So from currents, we have

$$N = \frac{R+a+b}{b}.$$

This is one equation; and it enables the unknown resistances a and b to be found in terms of R and N when it is taken in conjunction with

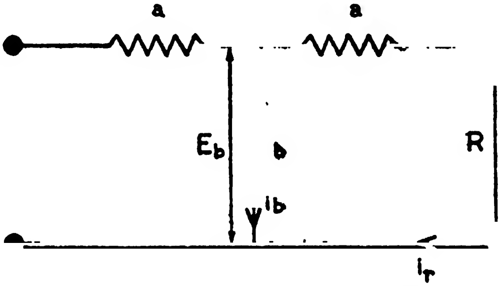


FIG. 3. THE "T" ATTENUATOR

another equation, and so the network may be built up. Consider voltages now in order to get the other equation.

The received voltage is $E_r = Ri_r$, while the sent voltage E_s is that across b, together with that lost in the first resistance a.

The voltage ratio now becomes

$$\frac{E_s}{E_r} = \frac{(R+a)i_r + a\dfrac{(R+a+b)}{b}i_r}{Ri_r} = N$$

So $$N = \frac{(R+a)b + aR + a^2 + ab}{Rb}.$$

Using the other equation

$$N = \frac{R+a+b}{b}$$

which was derived from the currents, makes $(R + a)\,b + aR + a^2 + ab = R\,(R + a + b)$ by equating the two.

From this

$$R^2 = 2\,ab + a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is an all-important formula in ladder network theory. It is the gateway to the study of ladder filters. It means that the characteristic impedance of a ladder network built up of two a's and a b is given by $R = \sqrt{2\,ab + a^2}$. This will be used later in filter design. Meanwhile we desire to find a and b.

Now $b = \dfrac{R^2 - a^2}{2a}$ from equation (1) above.

If this is put in

$$N = \frac{R + a + b}{b}$$

the result is

$$N = \frac{R + a}{R - a}$$

or $RN - aN = R + a$
which gives the resistance a as

$$a = R\left(\frac{N - 1}{N + 1}\right)$$

Notice that as the current ratio N becomes very large $a = R$ nearly, in other words for 600^{Ω} line, say, $a = 600^{\Omega}$ when the attenuation is large. To find b use the result

$$b = \frac{R^2 - a^2}{2a} \text{ and put } a \text{ in it.}$$

Then
$$b = \frac{R^2 - R^2\left(\dfrac{N - 1}{N + 1}\right)^2}{2R\left(\dfrac{N - 1}{N + 1}\right)}$$

$$= \frac{R^2\,(N + 1)^2 - R^2\,(N - 1)^2}{2R\,(N - 1)\,(N + 1)} = \frac{R\,4N}{2(N^2 - 1)}$$

or
$$b = \frac{R2N}{N^2 - 1} = 2R\left\{\frac{N}{N^2 - 1}\right\}$$

Here as N becomes larger b becomes smaller; and if N approaches 1, i.e. no loss, then b becomes very large. The following table gives the value of a and b for a "T" attenuator of this type.

TABLE 2

THE VALUES* OF THE RESISTANCES IN THE "T" ATTENUATOR

(Two a's and one b in shunt, Fig. 3)

db	a	b
	Ohms	Ohms
1	34·4	5190
2	68·8	2583
3	102·6	1704
4	135·8	1258
5	168	986·9
6	199·2	803·1
7	229·4	670
8	258·2	568
9	285·6	487
10	311·6	422
15	418·8	220·5
20	490·6	121·3
25	536	67
30	562	38
35	578	20·6
40	587	12
45	594	6·78
50	596	3·8
55	598	2·13
60	599	1·2
70	600	0·38
80	600	0·12
90	600	0·038
100	600	0·012

The Resistance Values for a "T" Attenuator

Each of the series arms of the "T" is $R\left(\frac{N-1}{N+1}\right)$.

The shunt arm is $2R\left(\frac{N}{N^2-1}\right)$.

These two formulas make it possible to design "T" attenuators for any impedance R and any attenuation. N is the voltage ratio, which must be calculated from the decibel desired attenuation figure. A network may be made up to have an impedance of, say, 600 ohms looking in one way when, say, 150 ohms is connected to the far terminals, and looking like 150 ohms when the first terminals are connected to a 600-ohm line. Thus the "pad" connects together

* These are slide rule results.

a 600- and a 150-ohm line with correct matching of the impedances. This is a separate calculation from that just given.

Fig. 4 shows a circuit for a 0–10 db attenuator, and Fig. 5 one for 0–100 db. The first box uses only two switch levels and 30 resistances, while in the second case an extra switch level enables a saving

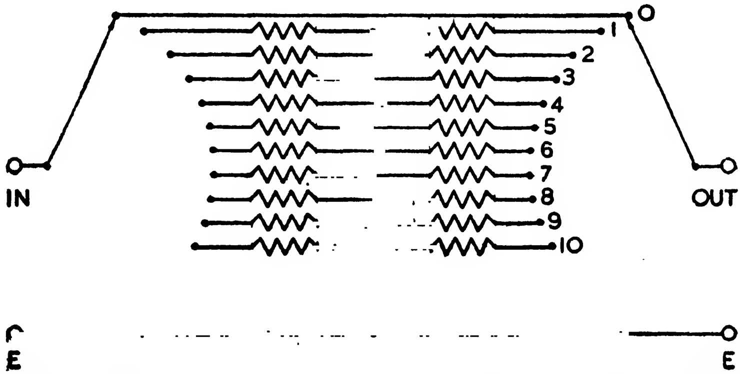


FIG. 4. "T" ATTENUATOR 0–10 DB, SHOWING SWITCHING ARRANGEMENT (30 COILS ARE NEEDED)

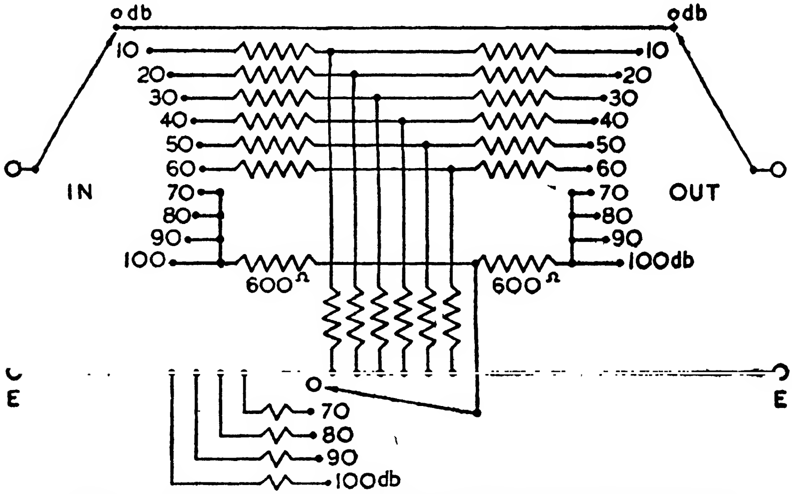


FIG. 5. "T" ATTENUATOR 0–100 DB, SHOWING SWITCHING CIRCUIT (24 COILS ARE NEEDED)

of six series resistances to be effected because on the higher values the a's are very nearly equal to 600^{Ω} while the shunt b alone needs to be altered.

In the design of filters the full arm of the top of the "T," which is twice *a*, is called Z_1, and this makes it like Fig. 6.

The next thing is to decide how the coils should be wound.

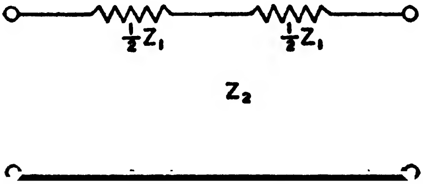


FIG. 6. A NUMERICAL EXAMPLE OF AN ATTENUATOR—FOR A 25Ω MICROPHONE

db	$\frac{1}{2}Z_1$	Z_2
1	1·5	216
2	2·9	108
3	4·3	71
4	5·6	52
5	7·0	41

Resistance Coil Winding

If resistance coils are just wound like a reel of cotton, the inductance may spoil the result. If, however, they are wound with a loop of wire as shown in Fig. 7 the inductive effect vanishes, because current

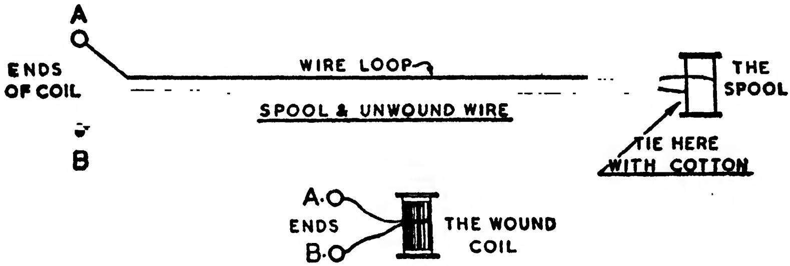


FIG. 7. WINDING NON-INDUCTIVE SPOOLS

goes round one way to the inner end of the loop and then back again. Unfortunately, capacity is now introduced. Wound in this second way the spool somewhat resembles a transmission line with resistance and capacity, short-circuited at the far end. If the wire be of R ohms per inch loop and if C farads is the capacity between the two wires of the loop per inch, the impedance of the spool is $Z_0 \tanh Pl$, and this becomes $\sqrt{\frac{R}{C\omega}} \tanh l\sqrt{RC\omega}$

Expanding $\tanh Pl$ as $Pl - \frac{(Pl)^3}{3} +$ negligible terms and writing Z_0 as $\frac{Rl}{Pl}$ makes the coil impedance

$$Rl\left(1 - \left(\frac{Pl}{3}\right)^2\right), \text{ which is } Rl\left(1 - \frac{1}{3}RlCl\omega\right)$$

Here Rl is the ohmic resistance of the loop and Cl the total capacity, say C_t.

Hence

$$\frac{\text{impedance}}{\text{ohmic resistance}} = 1 - \frac{(\text{ohmic res.})\ (C_t)\omega}{3}$$

This fraction $$\frac{(\text{ohmic res.})\ (C_t)\omega}{3}$$

is the fraction by which the coil is in error. It is a vector at right angles as it carries by right a j.

A formula by Cour and Bragstad gives the capacity of the wire used in the following calculation as $24\mu\mu$F. per yard loop. The resistance was 100 ohms per yard of wire, giving a loop 5 yards long for a 1000^{Ω} coil, i.e. $C_t = 120\mu\mu$F.

Then at 800c/s $Pl = 0{\cdot}0245$

Therefore $\frac{(Pl)^2}{3} = 0{\cdot}0002$, giving an impedance with a j carrying 0·02 per cent of the current. At 3200 cycles $Pl = 0{\cdot}049$, making $\frac{(Pl)^2}{3} = 0{\cdot}0008$, i.e. an impedance error with a j carrying 0·0008 or 0·08 per cent of the main current.

It must be remarked that if an impedance has a small j term added, its total size is practically unaltered, so if in the network which is being built up with the coils phases are not in question, but merely attenuation, this coil is excellent; for it will be right to a few parts in ten thousand as far as the error due to capacity is concerned.

The wire used by the author here, however, is a convenient size for getting over this capacity error. If too large, too much wire is needed, the length of the loop l is too large, and a big error comes in. If too small a wire is used, errors in soldering the ends become too big.

A still better way is to wind with a single wire, using half the wire up, and then wind alongside or on top with more of the wire in the opposite direction, but the above is very good.

When winding small values of resistance the error in joining up the end becomes large, and if two or more wires are wound in parallel these may be wound in opposite directions and one wire only varied to get the right adjustment. By using n wires in parallel the accuracy is increased n^2 times.

Winding Low Resistances

It is much easier to make an accurate low resistance if the thick wire, made a bit too long, is shunted by a long thin wire and this altered to bring the resistance down, as shown in Fig. 8.

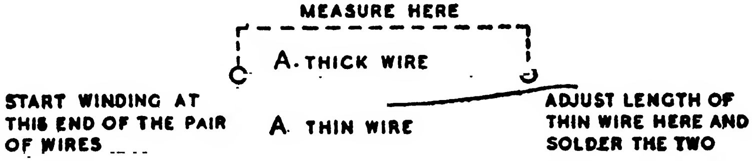


FIG. 8. ACCURATE WINDING OF LOW RESISTANCES MADE EASY

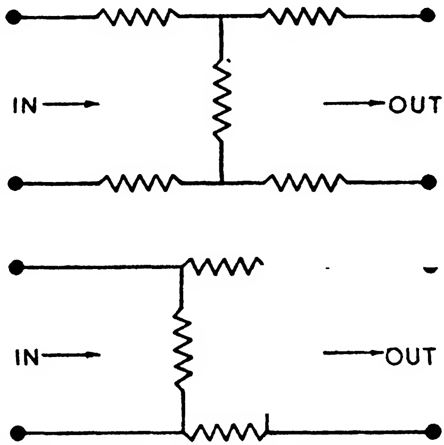


FIG. 9. DOUBLE "T" AND "Π" NETWORKS

Networks Balanced to Earth

When a network balanced to earth is required, the double Π network requires only four resistances instead of five for a double "T" as shown in Fig. 9.

CHAPTER II

LADDER FILTERS

In accordance with Theorem IV in Chapter IX any collection of reactances will form a filter. In this case, however, many odd pass and attenuation bands may come in at various places in the frequency scale. It is, therefore, necessary to employ design methods in order to make the edges of the pass band come at the desired frequency and to obtain the desired impedance, not to speak of the desired attenuation.

Historically it was the invention of the coil-loaded telephone cable by Pupin that constituted the first filter.

It was seen that the inductance lumped in the coils with capacity between coils gave the filtering action.

Logically, the proper way to begin is to study various circuits, such as the ladder and lattice, using resistances first; and, then,

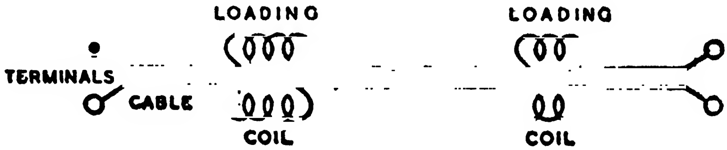


FIG. 10. COIL-LOADED CABLE

after building up a complete theory of such networks, put reactances in the arms of each to make a filter. If this were done, the reader might find it hard to see where the text was leading to, and therefore difficult to follow, so the ladder filter is treated first of all, for its own sake, and as an example of the method to be followed in lattice and other filter circuits.

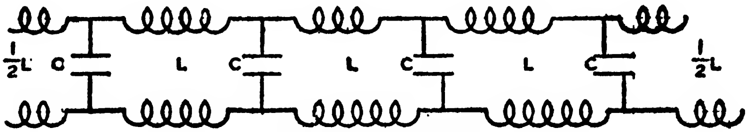
The coil-loaded telephone cable with its cut-off frequency phenomena constituted a low pass ladder filter, shown in its simplest form in Fig. 10. Such filters need a special end coil or condenser.

There are two ways of arranging the termination of a ladder filter; one is the **half-series termination** where the end component consists of a coil of half the inductance of the main ones, i.e. half the series impedance of the ladder (see Fig. 11 (*a*)).

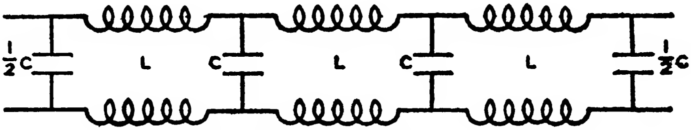
The other is the **double-shunt termination**, which in this case means a condenser of half the full capacity. In other words, the

impedance of the first shunt with this termination is double the ordinary impedance of the other rungs of the ladder (see Fig. 11 (*b*)).

In order to obtain definite formulas for ladder circuits we go back to Chapter I on attenuators, where it was shown that the characteristic impedance of a ladder network is $\sqrt{2ab + a^2}$. Here a is the value of one arm of the "T." When several "T's" are joined



(*a*) Half-series termination.



(*b*) Double-shunt termination.

FIG. 11. LOW PASS FILTERS EQUIVALENT TO COIL-LOADED CABLE

up to make a ladder it is convenient to take an a covering all the series impedance between two rungs, i.e. twice as big as that used in working out attenuators.

This makes the characteristic impedance $Z_0 = \sqrt{ab + \frac{a^2}{4}}$ for the ladder. This is applied to a ladder built up of "T's," so that the end section has a series impedance $\frac{1}{2}a$ where a is the total impedance between two rungs. If it is half in each line to secure a balance to earth, a is then the total of these halves. Thus this formula applies to a ladder network which is *half-series terminated*.

The impedance of the double-shunt terminated ladder may be calculated from this formula for the half-series terminated ladder by adding an extra $\frac{1}{2}a$ in series followed by $2b$ in parallel. The result is that the general impedance formula for a *double-shunt terminated ladder* is

$$Z_0 = \frac{\sqrt{ab}}{\sqrt{1 + \frac{a}{4b}}}$$

Let us consider for a moment what these formulas mean. They have been derived on the assumption that the arms of the ladder

are pure resistances. One may ask "Is this all right? Can we suppose or assume resistances when we know the arms are not going to be resistances, derive a general formula on that assumption, and then proceed to put $jL\omega$'s and $\frac{1}{jC\omega\text{'s}}$ in for coils and condensers?"

This question is fundamental. The answer is that we can do so, and one may argue backwards from the j notation somewhat as follows. In using the j notation we put reactances in series and in parallel just as if they were resistances. In some textbooks Z_1 and Z_2 are used for impedances, but if they are in series it is $Z_1 + Z_2$, just like two resistances. Why not then use a and b, easy letters, for plain resistances? It makes it simpler to think about. The general formulas

$$\sqrt{ab}\sqrt{1 + \frac{a}{4b}} \text{ and } \frac{\sqrt{ab}}{\sqrt{1 + \frac{a}{4b}}}$$

apply to any ladder, but each particular circuit must be worked out separately, and various impedance frequency curves will result from different circuits.

There is an interesting peculiarity about these two formulas. They multiply to ab. Although a and b are usually both imaginary quantities, because they are both reactances which vary with frequency, ab is often a real figure, and may even be a constant, independent of frequency, and if it be positive, then a square root gives a real quantity too.

Campbell's theorem connecting characteristic impedance with attenuation, which is a corollary of the author's Theorem IV (see Chapter IX), is as follows—

"When the characteristic impedance is unreal the filter attenuates, but when, i.e. at another frequency, the impedance is real, the filter has a pass band."

So if one studies the characteristic impedance of a particular circuit one not only knows how it varies with frequency in *size* and *reality*, but by noticing where it is unreal one finds the attenuation bands, i.e. where they come in the frequency scale.

Take the half-series case first, and work it out as if a and b were pure resistances.

Then $$Z_0 = \sqrt{ab + \frac{a^2}{4}} = \sqrt{ab}\sqrt{1 + \frac{a}{4b}}$$

The reactances a and b must now be put in. Take the low pass filter for a start: a is a coil and b a condenser, so

$$a = jL\omega \text{ and } b = \frac{1}{jC\omega}$$

Then $\sqrt{ab} = \sqrt{\frac{L}{C}}$, which is a numeric, and $Z_0 = \sqrt{\frac{L}{C}} \cdot \sqrt{1 - \frac{LC\omega^2}{4}}$

This is real when ω is less than $\frac{2}{\sqrt{LC}}$ and reactive when ω exceeds this value; so, without more ado, the cut-off frequency is $\frac{1}{\pi\sqrt{LC}}$ and there is attenuation above this frequency, but none below. The amount of the attenuation is another matter. When $\omega = 0$, i.e. at very low frequencies, $Z_0 = \sqrt{\frac{L}{C}}$, but the impedance in the pass band falls as ω increases and is 0 at the cut-off. When such a filter is said to have a nominal impedance of, say, 600^Ω, this is the value of $\sqrt{\frac{L}{C}}$. The filter is usually designed so that $\sqrt{\frac{L}{C}}$ is 600^Ω if the filter is to work into a load of 600^Ω. There is then a lack of matching, which becomes worse for frequencies nearer the cut-off than it is for lower frequencies. (See Fig. 12.)

This lack of matching the ends, together with phase changes in the filter, causes a loss, even in the pass band, but this loss is not usually serious. The formula

$$Z_0 = \sqrt{\frac{L}{C}}\sqrt{1 - \frac{LC\omega^2}{4}}$$

if plotted as Z_0 against ω gives a "quarter-circle" diagram. The line to which the filter is connected may be called R ohms, and this is the value of $\sqrt{\frac{L}{C}}$,

so
$$Z_0 = R\sqrt{1 - \frac{LC\omega^2}{4}}$$

Further, as

$$f_0 = \frac{1}{\pi\sqrt{LC}}$$

where f_0 is the cut-off frequency, the LC in the formula for Z_0 may be

put in terms of the frequency f being used and f_0 the cut-off frequency. This makes

$$\frac{Z_0}{R} = \sqrt{1-x^2} \text{ where } x = \frac{\text{frequency}}{\text{cut-off frequency}}$$

and shows the equation to be a circle. As

$$\sqrt{\frac{L}{C}} = R \text{ and as } f_0 = \frac{1}{\pi\sqrt{LC}}$$

design may be carried out at once. Multiplication gives

$$L = \frac{R}{\pi f_0} \text{ and division } C = \frac{1}{\pi R f_0}$$

These two formulas are very simple and highly important. They are a milestone and enable one to make a low pass filter.

In particular, the expression

$$\frac{Z_0}{R} = \sqrt{1-x^2}$$

is important because it shows that

(1) the impedance varies with frequency;

(2) for low frequencies the impedance Z_0 of the filter is nearly equal to a constant value R because a low frequency or small x makes $\sqrt{1-x^2}$ nearly 1, and so $Z_0 = R$ nearly;

(3) at some frequency given by x having the value 1 the impedance is zero;

(4) the impedance is real when x is less than 1 since $1 - x^2$ is less than 1;

(5) the impedance is imaginary when x is above 1 because $1 - x^2$ is now negative and its root is unreal;

(6) for high values of x, the impedance is high since $\frac{Z_0}{R}$ is then nearly equal to x.

Curves will be given later (see Fig. 12).

Formulas for Components of Simple Ladder Low Pass Filter

The coil $L = \frac{R}{\pi f}$

The condenser $C = \frac{1}{\pi R f}$

EXAMPLE

Design a filter for a 1000-ohm line to cut off all frequencies above 500 cycles.

Then $$L = \frac{1000}{500\pi} = \frac{2}{\pi} \text{ henrys}$$

And $$C = \frac{1}{500\pi \times 1000} \text{ farads, or } \frac{2}{\pi} \text{ microfarads}$$

These two formulas are fairly easy to remember. The easiest way is to work out a few examples made up by oneself. Then, using rough and ready arithmetic (let π be three, for once), work out a

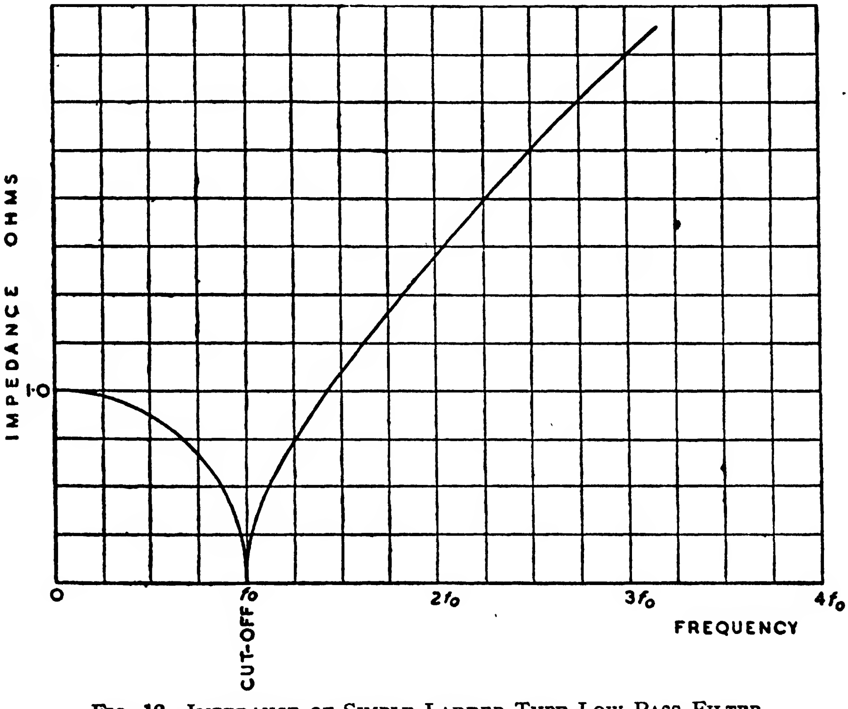


FIG. 12. IMPEDANCE OF SIMPLE LADDER TYPE LOW PASS FILTER (HALF-SERIES TERMINATED)

few cases mentally. When one knows the coil and condenser sizes one can build the filter. The two formulas form the basis of the formulas for all the more important ladder filters.

One might reasonably want to calculate attenuations at once, but we are going to work out the characteristic impedances first. There is much to be said for doing so. If one goes to buy an electric

iron or a lamp, one wants to know the current it takes. Impedance tells this for different frequencies.

The attenuation gives the current at any place along the filter. Impedance gives the current entering. Both assume an infinitely long filter to avoid reflections from the distant end.

Impedance of Low Pass Filter

The impedance in the pass band is $\sqrt{1-x^2}$ times the nominal R, which is $\sqrt{\frac{L}{C}}$. Using this R value as a unit then, it follows that the impedance in the pass band of a half-series terminated low pass plain ladder filter is a variable resistance when the frequency is varied, following a graph which is just a quadrant of a circle (see Fig. 12). Knowing the line for which it is desired to make a filter settles the value R ohms. The cut-off is the only other factor. These with π give L and C. The L is the total inductance between two condensers. If it is desired to put half of it in each side of the ladder, double-wound coils each of total inductance L must be used.

Half-series Case

The low pass filter if half-series ended falls to zero at cut-off along a quadrant of a circle. This is in the pass band because at frequencies below cut-off the x in $\sqrt{1-x^2}$ is a fraction, reducing $1-x^2$ as it gets bigger with rising frequency.

In the attenuation band we have $\sqrt{x^2-1}$, which is turned round because x^2 is greater than 1 since x is so, and it is necessary to take out -1, giving j when the root of minus one is taken. The impedance is now a reactance.

Let us consider the characteristic impedance of this filter in greater detail. The calculation is given in Table 3.

$$Z_0 = \sqrt{\frac{L}{C} - \frac{L^2\omega^2}{4}}$$

for the half coil ended filter. When ω is low, this is real, i.e. a pure ohmic resistance of value $\sqrt{\frac{L}{C}}$ for low frequencies, falling to zero at a frequency given by

$$\frac{L}{C} = \frac{L^2\omega^2}{4}, \text{ i.e. } \frac{2}{\omega_0} = \sqrt{LC}$$

Also note that *above* this *frequency* there is a *minus sign* with its *root*

a j indicating a pure reactive impedance, rising now with frequency. (See Fig. 12.)

TABLE 3

IMPEDANCE OF LOW PASS SIMPLE LADDER FILTER ABOVE CUT-OFF (HALF-SERIES TERMINATED)

x	$\sqrt{x^2 - 1}$
1·05	0·353
1·1	0·458
1·2	0·663
1·3	0·83
1·4	0·98
1·5	1·12
1·6	1·25
1·7	1·37
1·8	1·5
1·9	1·61
2·0	1·732
2·5	2·29
3·0	2·8
3·5	3·34
4·0	3·87

Here x is again the frequency as a multiple of the cut-off frequency. The figures worked out are impedance, as a multiple of the nominal impedance of the filter. The curve is therefore general for all filters with the same circuit.

If, for example, a filter with 1000 ohms nominal impedance is worked at 10 kc., its cut-off being 5 kc., and it is a low pass filter terminated in *half a coil*, the impedance is found at 10 kc. from the value 1·732 opposite 2 in the table because 5 kc. and 10 kc. make $x = 2$. The 1·732 must be multiplied by the nominal 1000, the result being 1732 ohms.

These remarks apply to the simple half-series ended filter. If it is ended in this way the coil at the end must have half the calculated inductance as given by the formula $L = \frac{R}{\pi f}$, which gives the whole coil.

If the filter is double-shunt ended the ends must be half value condensers.

Double-shunt Case

This makes the impedance *rise* in the pass band to infinity at cut-off because the formula becomes $\frac{1}{\sqrt{1 - x^2}}$ according to Table 4.

TABLE 4

RISING IMPEDANCE IN THE PASS BAND
(LOW PASS FILTER DOUBLE-SHUNT ENDED)

x	Impedance
0·1	1·008
0·2	1·023
0·3	1·049
0·4	1·094
0·5	1·157
0·6	1·25
0·7	1·4
0·8	1·67
0·9	2·2
1·0	Infinity

(See also Fig. 13.)

These values are merely the reciprocals of the values for the quadrant of the circle which falls to zero (Fig. 12).

The values in the attenuation band for the **double-shunt ended** low pass filter are as follows.

The formula is $\frac{1}{\sqrt{x^2-1}}$.

TABLE 5

IMPEDANCE IN THE ATTENUATION BAND
(LOW PASS FILTER DOUBLE-SHUNT ENDED)

x	Impedance
1	Infinity
1·1	2·19
1·2	1·51
1·3	1·20
1·4	1·025
1·5	0·896
2	0·578
2·5	0·436
3	0·353
3·5	0·298
4	0·259

(See also Fig. 14.)

Referring to Table 3 for the half-series case, as ω becomes large above the cut-off, Z_0, now reactive, becomes large too. This is

quite right because the first condenser is a short circuit at high frequencies, and one measures the end coil, which is $\frac{1}{2}j\omega$ only, which is what the formula reduces to when ω is big. If a

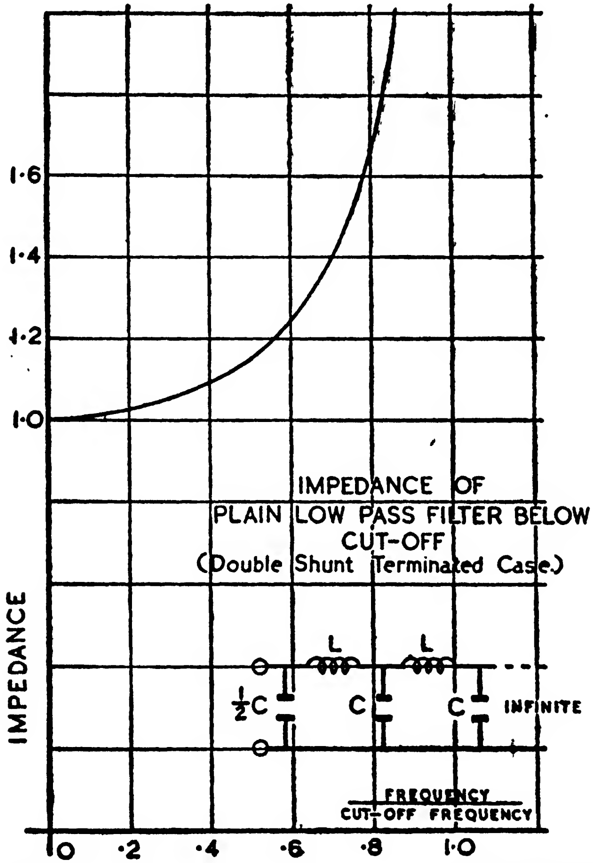


FIG. 13. IMPEDANCE OF LOW PASS FILTER BELOW CUT-OFF (DOUBLE-SHUNT TERMINATED)

double-shunt terminated filter is used, $\frac{Z_0}{R}$ is for all frequencies the reciprocal of what it is for the half coil terminated filter. That is to say, Z_0 differs for the two terminations.

The two cases are shown in Figs. 12, 13, and 14.

The above characteristics for the impedances of low pass filters

with either termination are of value in themselves and also for other reasons.

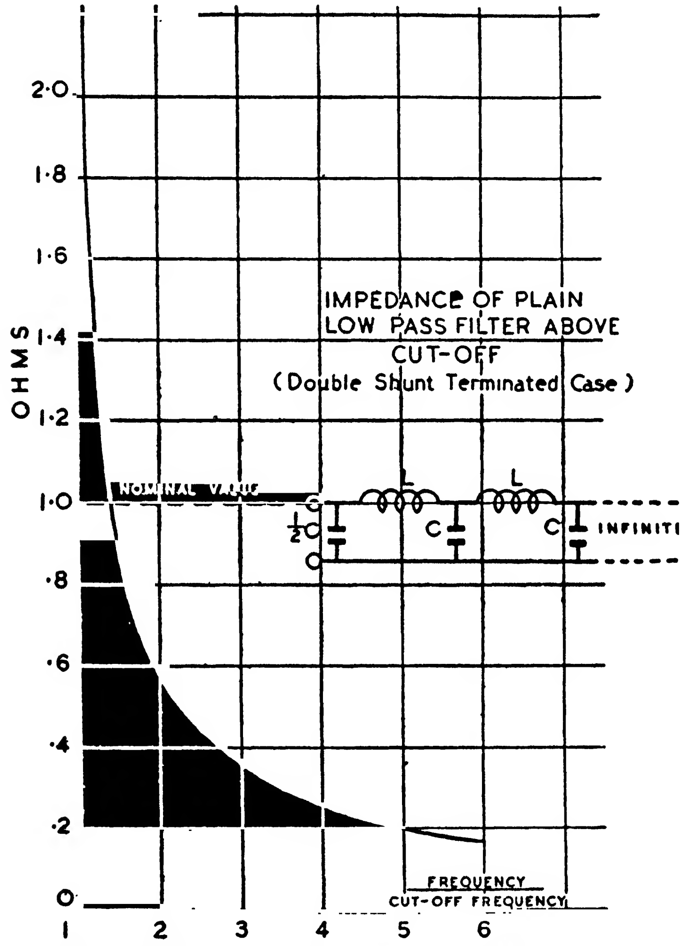


FIG. 14. IMPEDANCE OF LOW PASS FILTER ABOVE CUT-OFF (DOUBLE-SHUNT TERMINATED)

The Attenuation of the Low Pass Filter

The change of impedance of the infinitely long filter from real to imaginary at a certain critical frequency gives a clue to the filter action, for if any apparatus having two terminals measures as a resistance, the apparatus accepts energy, and as neither L nor C can

dissipate but only store energy it is highly probable that the *energy* is travelling along the filter, from section to section.

Further, in the case of a continuous wave from an oscillator, if there is not to be an infinite build up of energy near the end, the energy must be going in, travelling along the line as fast as it is supplied, suggesting no attenuation. On the other hand, at a frequency above the cut-off frequency, the reactance effect shows that no continuous supply of energy is being accepted, but that there is just a surging of energy into and out of the filter at each half cycle. This suggests attenuation.

To find the attenuation, it is convenient to measure, not the mere ratio of current entering and leaving a section, but its logarithm—because current ratios multiply, one after the other, but logarithms merely add up. The current ratio entering and leaving a ladder, in terms of the series and shunt arms a and b is rather complicated, but, curiously enough, if the current entering a section is called i_1 and the current leaving the section is called i_2, the ratio being $\frac{i_1}{i_2}$, although the *ratio* depends on the series and shunt impedances a and b of the ladder in a *complicated* way, the expression for

$$\frac{1}{2}\left\{\frac{i_1}{i_2}+\frac{i_2}{i_1}\right\}$$

in terms of a and b is very simple: it is just

$$\left\{1+\frac{a}{2b}\right\}$$

The first expression will be recognized as

$$\frac{\varepsilon^{P}+\varepsilon^{-P}}{2}$$

if P denotes the logarithm of the ratio of i_1 to i_2 (for ε^{-P} is the reciprocal of ε^{P}, so if ε^{P} is i_1 over i_2, then ε^{-P} is i_2 over i_1). This expression

$$\tfrac{1}{2}\{\varepsilon^{P}-\varepsilon^{-P}\}$$

has a name in *hyperbolic* trigonometry. It is called cosh P.

Formula for the Attenuation of a Ladder Filter

The above formulas give

$$\text{Cosh } P = 1+\frac{a}{2b}$$

This is a very important formula, because it gives the attenuation (which is the logarithm of a current ratio, so that when P is known the current ratio is also known) in terms of the arms a and b of the ladder filter. What we have not done is to prove that

$$\frac{1}{2}\left\{\frac{i_1}{i_2} + \frac{i_2}{i_1}\right\}$$

will give the simple form

$$1 + \frac{a}{2b}$$

in the case of a ladder. Also, having the formula, one needs to know how to use it.

Proof of the Formula for the Propagation Constant of Ladder Filters (Based on the Calculation of Half-section Attenuation)

Take half a section, with its series arm as in Fig. 15. Fig. 16 shows an example of Fig. 15 for the case of a low pass filter.

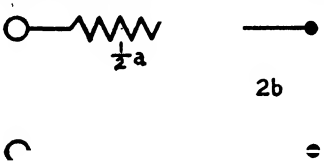


FIG. 15. HALF A SECTION

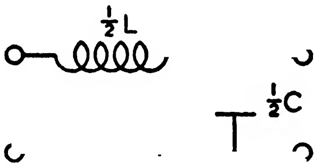


FIG. 16. HALF A SECTION OF A LADDER LOW PASS FILTER

The well-known cable formula for the impedance of a line opened and closed applies. Thus

$$Z \text{ closed} = Z_0 \tanh \tfrac{1}{2}P = \tfrac{1}{2}a \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Calling the attenuation of a complete section P

$$Z \text{ open} = \frac{Z_0}{\tanh \frac{1}{2}P} = \tfrac{1}{2}a + 2b \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiply these two equations together.

$$Z_0^2 = \tfrac{1}{2}a\,(\tfrac{1}{2}a + 2b) = ab + \frac{a^2}{4}$$

which gives the characteristic impedance and so immediately proves this formula. To find the attenuation is fairly easy. Dividing the two equations (1) and (2), we have

$$\text{Tanh}^2\,\tfrac{1}{2}P = \frac{\frac{1}{2}a}{\frac{1}{2}a + 2b} = \frac{a}{a + 4b}$$

Our aim is to get a simple formula for cosh P, as the formula for cosh P is the simplest in the case of ladder structures. Express tanh as sinh divided by cosh.

$$\frac{\sinh^2\frac{1}{2}P}{\cosh^2\frac{1}{2}P} = \frac{\cosh^2\frac{1}{2}P - 1}{\cosh^2\frac{1}{2}P} = 1 - \frac{1}{\cosh^2\frac{1}{2}P} = \frac{a}{a + 4b}$$

So

$$\frac{1}{\cosh^2\frac{1}{2}P} = 1 - \frac{a}{(a + 4b)}$$

The next thing is to turn this upside down,

$$\cosh^2\tfrac{1}{2}P = 1 + \frac{a}{4b}.$$

But $\cosh P = 2\,(\cosh^2\frac{1}{2}P) - 1$.

This is like the easy formula of half an angle in circular trigonometry. So

$$\cosh P = 2 + \frac{a}{2b} - 1$$

The final formula then is

$$\cosh P = 1 + \frac{a}{2b}$$

This proves the formula for the propagation constant in an easy way. When both a and b are known in the j notation, $\frac{a}{2b}$ is known, cosh P is known, and P can be found from tables of cosh P.

In this proof everything depends on the "open" and "closed" impedance formulas, which are so much used by cable manufacturers when they have made a piece of cable and wish to find its characteristic impedance by open and closed tests.

The proof to the cable formulas follows. It is a great benefit to treat filters as cables and use the transmission methods.

An Easy Proof to the Impedance of a Length of Line Open and Closed at the Far End

This is the easiest way to prove these formulas. Draw a picture of the real line cut at the place desired. Call the attenuation P up to that point.

The oscillator may be thought of as having the same impedance as the infinite line. Call this Z_0.

Let the oscillator put a voltage E on the sending end of the infinite line. Then let the line be cut. The voltages and currents change.

In a D.C. case the currents and voltages would change in this sense, the currents would be less, and the voltages greater everywhere when the line was cut.

In an A.C. case resonance would occur in some cases and these decreased currents might prove increases and so on. The j notation takes care of all that, so one can work it out as a D.C. case. The trick is to introduce the infinite line with an extra generator in it, which must make the current zero if we want it to be like a line which is cut.

Here then is the line and an infinite line with an extra generator (see Fig. 17).

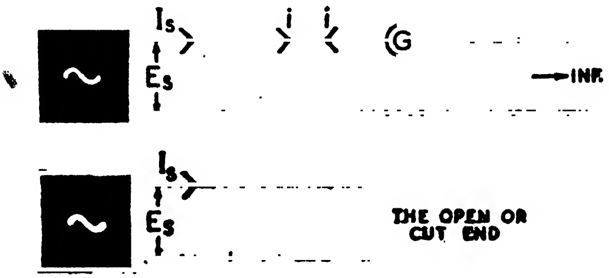


FIG. 17. "OPEN" CABLE FORMULA

Without the added generator, G, the sent current is E, the voltage the oscillator puts on the infinite line divided by Z_0 its characteristic impedance. The current i at G is

$$\frac{E}{Z_0}\varepsilon^{-P}$$

because ε^{-P} is the reduction factor.

If we are to simulate the cut line, the generator G must make the current at G zero. This means the generator G must put into the line a reverse current of strength

$$\frac{E}{Z_0}\varepsilon^{-P}$$

This will return to the sending end, and as the oscillator is supposed of impedance to match the line, there will be no reflections when the return wave arrives at the oscillator. The current wave returned to the oscillator is attenuated on its way back, so its value is

$$\frac{E}{Z_0}\varepsilon^{-2P}$$

The current leaving the oscillator is

$$\frac{E}{Z_0} - \frac{E}{Z_0}\varepsilon^{-2P}$$

because the go and return waves subtract. The return wave puts a pressure across the oscillator terminals, a rise of value decided by Z_0. It is

$$Z_0\frac{E}{Z_0}\varepsilon^{-2P} = E\varepsilon^{-2P}$$

Therefore, cutting the line raises the voltage E to $E + E\varepsilon^{-2P}$ and lowers the current from

$$\frac{E}{Z_0} \text{ to } \frac{E}{Z_0} - \frac{E}{Z_0}\varepsilon^{-2P}.$$

The new current

$$\frac{E}{Z_0}(1 - \varepsilon^{-2P})$$

is the I_s, the true sent current of the cut line. The new voltage $E(1 + \varepsilon^{-2P})$ is the E_s, the true sending voltage in the case of the cut line. Divide these, and it gives the impedance Z open.

$$Z_{op} = \frac{E(1 + \varepsilon^{-2P})}{\frac{E}{Z_0}(1 - \varepsilon^{-2P})}$$

This reduces to

$$Z_0\frac{\varepsilon^{P} + \varepsilon^{-P}}{\varepsilon^{P} - \varepsilon^{-P}}$$

by simply multiplying by ε^P. The result is $Z_0 \coth P$ because

$$\frac{\varepsilon^{P} - \varepsilon^{-P}}{\varepsilon^{P} + \varepsilon^{-P}}$$

is tanh P. This proves the formula

$$Z_{op} = Z_0 \coth P$$

The next thing is to consider the cable short-circuited at the far end (see Fig. 18).

Cable Short-circuited

In order to make the uncut line like the line which is cut and short-circuited, one may connect a generator of voltage across the line helping current to flow across from positive to negative. The result,

if the generator reduces the voltage across the line and across its own terminals to zero, is to make the current double. It is like connecting two cells in series short-circuited. If their voltages and impedances

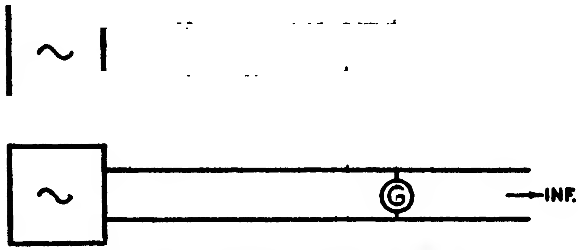


FIG. 18. "CLOSED" CABLE FORMULA

are equal, they must produce zero voltage across the terminals when connected. This generator, therefore, introduces or superimposes the same terminal voltage and current that the other one does. The voltage is added in such a direction as to reduce the voltage everywhere (it is reduced to zero at the short circuit) and increase the current everywhere. The result is that the sent current is in this case

$$\frac{E}{Z_0}(1 + \varepsilon^{-2P})$$

and the sending voltage is $E\,(1 - \varepsilon^{-2P})$.

This makes

$$\frac{\text{Sending voltage}}{\text{Sent current}} = Z_0\left\{\frac{\varepsilon^P - \varepsilon^{-P}}{\varepsilon^P + \varepsilon^{-P}}\right\} \text{ which is } Z_0 \tanh P$$

These, then, are the two formulas for the impedance of a cable with the end open and closed.

The two formulas $Z_{op} = Z_0 \coth P$

and $Z_{cl} = Z_0 \tanh P$

tell the impedances of the two cases. In the past one seldom wished to use a cable with the end shorted, though we do so now in radio work and get resonance.

What was and is still valuable, however, is to measure Z open and Z closed on a bridge and multiply the two. When this is done $\tanh P$ cancels $\coth P$ and $Z_{op}\, Z_{cl} = Z_0^2$, which immediately finds Z_0, the characteristic impedance.

Division gives $\tanh P$, from which the propagation term P is found.

The great feature of these formulas is that they enable any circuit

such as a ladder, lattice, etc., to be taken and analysed to give general formulas for its Z_0 and its propagation.

The Use of the Attenuation Formula

When a and b are one negative and the other a positive reactance, i.e. $-j$ and j, the result is $(1 - a \text{ real})$, which is a real fraction until

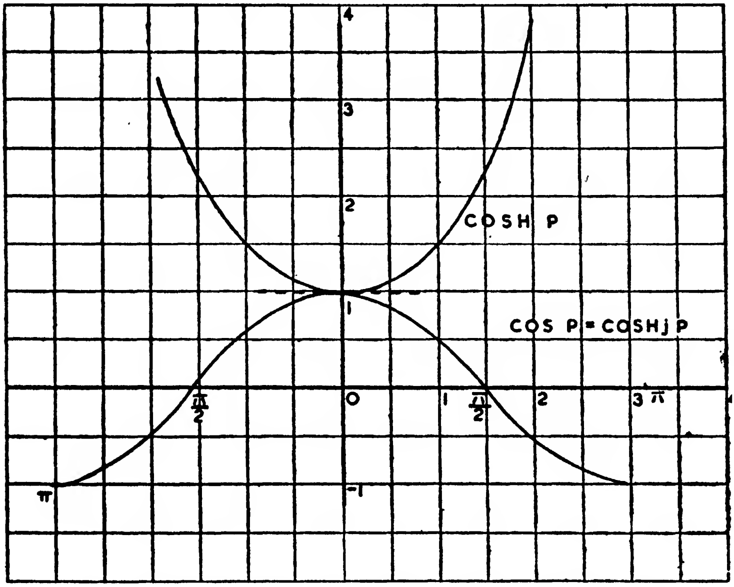


FIG. 19. COSH P WITH REAL AND UNREAL VALUES OF P

$\frac{1}{2}a/b$ becomes more negative than -1. Until then it is fractional: first positive, then negative. When cosh P is fractional, P is unreal; and $\cos B = 1 + \frac{1}{2}\frac{a}{b}$, putting B for unreal propagation constant, which is a phase shift of B degrees.

Filters in their pass bands, then, give a phase shift, section by section. A negative cosine means a shift of angle greater than 90°. When, however, cosh P is real and greater than unity (without regard to its + or − sign), then P is real, which means so much attenuation in népers.

The attenuation begins when cosh P is 1, that is when $\frac{a}{2b}$ is -2, which is when $\frac{a}{4b}$ is -1 in the case of this filter. Cosh P is drawn for P imaginary and P real in Fig. 19.

Cosh P can be negative since it is

$$\frac{\varepsilon^P + \varepsilon^{-P}}{2}$$

and ε^P is a current ratio. So is ε^{-P}, and if a current is reversed in phase with regard to another, their ratio is negative. This makes cosh P negative too, which is not at first obvious. It can also be seen by writing P as $A + jB$ and expanding cosh $(A + jB)$.

A Simple Explanation of the Action of a Low Pass Filter

If the filter is drawn with both coils and condensers divided into two, it becomes easy to see that there is no attenuation at the frequency of resonance of the half coil with the half condenser (this is

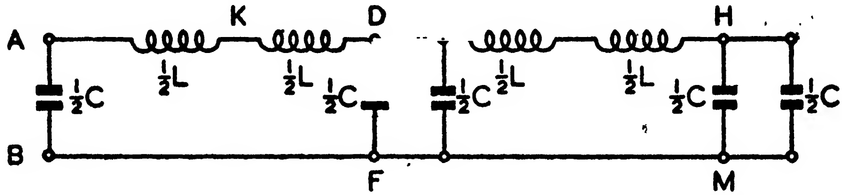


FIG. 20. THE CUT-OFF FREQUENCY OF THE LOW PASS FILTER

a definition of the cut-off frequency), but that there is attenuation above it (see Fig. 20).

At the frequency of resonance of the half coil $\frac{1}{2}L$ with the half condenser $\frac{1}{2}C$, the whole loop $ADFB$ is in resonance with the circulating current, and this section viewed alone has infinite impedance measured at the terminals AB, because the impedance round $ADFB$ is merely that of the $\frac{1}{2}L$ since the second $\frac{1}{2}L$ and $\frac{1}{2}C$ are zero in series resonance. Thus the ADF path is in anti-resonance with the $\frac{1}{2}C$ across A and B at a frequency f such that

$$\frac{1}{2} L\omega = \frac{2}{C\omega} \text{ or } \frac{2}{\omega} = \sqrt{LC}.$$

If then the loop $ADFB$ has an infinite impedance at the terminals AB and DF the same may be said of every other section. Thus the second section puts no shunt on the condenser between D and F, and the argument which applies to one isolated section is true of the whole chain when connected up. This proves the filter to have an infinite impedance at this frequency.

The impedance of the coil KD cancelling that of the condenser DF and leaving the coil AK only effectively in circuit means that

the total applied voltage is reproduced three times in the circuit *AKDF*, once across *AK*, once across *KD*, and once across *DF*. Thus the applied voltage is found at full volume across *DF*, the phase being obviously reversed, because the current is the same in all three elements of the *ADF* circuit, but a condenser voltage is in phase opposition to a coil voltage when the two are in series.

The voltage across *DF* being applied to the next section causes the same chain of events. The voltage at *DF* is found in full volume at *HM*, and so on. There is then **no attenuation**, but 180° phase change from section to section.

Suppose the argument repeated at a higher frequency. Again suppose the first section to be separated from the rest.

The impedance of the *ADF* path now exceeds that of the *AB* condenser, so the condenser takes more current and the section looks like a condenser as regards the current taken. Further, the voltage across *DF* is less than the voltage applied to *AB* now, for the impedance of coils rises, and that of condensers drops with frequency. There was equality before, now there is not. When the next section is coupled on at *DF*, it acts as a capacity load, increases the size of the $\frac{1}{2}C$ condenser between *D* and *F*, and so further lowers the voltage between *D* and *F*. The relative phases remain the same. Thus there is **attenuation** at any frequency above the

$$f = \frac{1}{\pi\sqrt{LC}}$$

and a steady phase change of 180° from section to section.

Phase Changes in Filters

Neglecting resistance in coils and condensers in the attenuation band as explained before, *P* is real and cosh *P* is found to be greater than unity, which is necessary, for cosh x starts from 1 when $x = 0$.

Cosh *P* may, however, be positive or negative. If positive there is no phase change from section to section. If, however, cosh *P* is found to be negative then there is a phase change of 180° from section to section, for it means that the propagation constant contains a real portion, say *A*, and an unreal, $j\pi$. Then cosh *P*, which is now cosh $(A + j\pi)$ is cosh $A \cos \pi + j \sinh A \sin \pi$ when expanded as cosh $(a + b)$, and as sin π is 0 and cos π is -1, the whole cosh $P = -\cosh A$, proving that a minus sign to the cosh *P* means a phase change of π. This proof is necessary because when cosh *P* turns out negative it does not follow as obvious that there is reversal of current, for cosh *P* is a complicated function but has that property.

This is given as a theorem by G. A. Campbell.

If the filter is all coils, cosh P is positive, and there is attenuation as long as both shunt and series arms behave as coils. (See Fig. 21.)

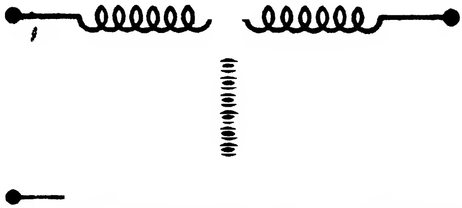


FIG. 21. THREE COILS ACTING AS A "T" ATTENUATOR

Notice particularly that attenuation depends on $\frac{a}{4b}$ being either positive or else negative but greater than 1. This is illustrated in Fig. 22.

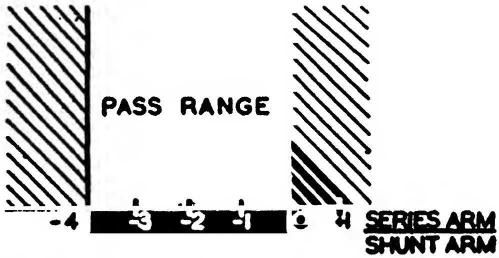


FIG. 22. ATTENUATION AND PASS RANGE OF LADDER FILTERS DEPENDING ON RATIO OF THE ARMS LYING BETWEEN 0 AND -4

The Attenuation Curve of the Low Pass Ladder Filter

$$\operatorname{Cosh} P = 1 + \frac{a}{2b}$$

Here $a = jL\omega,\ b = \frac{1}{jC\omega}$ and $\cosh\ P = 1 - \frac{LC\omega^2}{2} = 1 - 2\,x^2$

where $x = \frac{\text{applied frequency}}{\text{cut-off frequency}}$.

Attenuation begins at $x = 1$ and the result is as shown in Table 6.

The value of $B°$ from cos B is found by cosine tables, while the value of A from cosh A is found from tables of cosh x, for which see Fig. 19. A curve of the results is shown in Fig. 23.

TABLE 6

ATTENUATION CURVE OF THE SIMPLE LOW PASS LADDER FILTER AND THE PHASE CHANGE IN THE PASS BAND

Attenuation band			Pass band		
x	$1-2x^2$ (cosh P)	A or P	x	$1-2x^2$ cos B	$B°$
1·05	− 1·25	0·69	0	1·0	0°
1·1	− 1·42	0·89	0·1	0·98	11°
1·2	− 1·88	1·25	0·2	0·92	23°
1·3	− 2·38	1·51	0·3	0·82	35°
1·4	− 2·92	1·74	0·4	0·68	47°
1·5	− 3·5	1·93	0·5	0·5	60°
1·6	− 4·12	2·1	0·6	0·28	74°
1·7	− 4·78	2·25	0·7	0·02	88° 51′
1·8	− 5·48	2·39	0·8	− 0·28	106° 18′
1·9	− 6·22	2·52	0·9	− 0·62	128° 24′
2·0	− 7	2·65	0·95	− 0·805	143° 36′
2·5	− 11·5	3·14	1·0	− 1·0	180°
3·0	− 17	3·54			
3·5	− 23·5	3·85			
4·0	− 31	4·13			
4·5	− 39·5	4·38			
5·0	− 49	4·6			

Here A is in népers and is found from cosh A, while B is found from cos B in the previous column, using the appropriate table in each case.

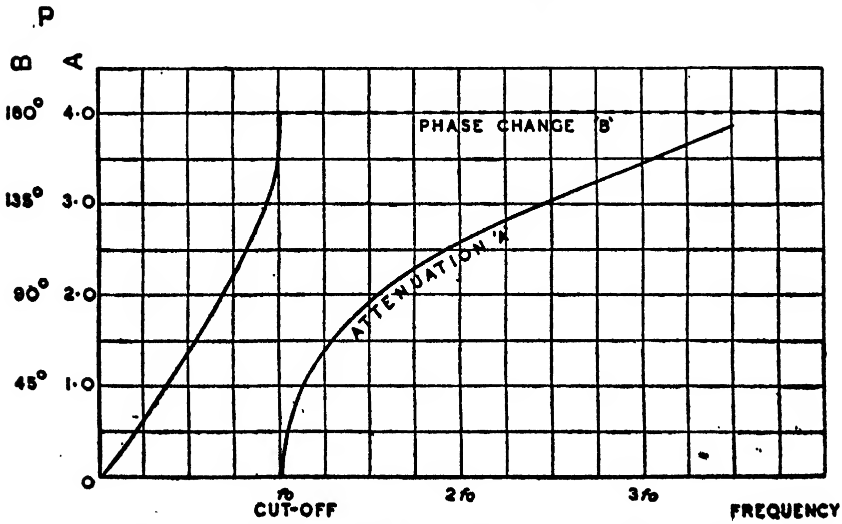


FIG. 23. PROPAGATION CONSTANT OF SIMPLE LADDER L.P. FILTER

The Use of " General " Curves

These curves are of fundamental importance, for they apply to the low pass filter. Since they are plotted for frequency divided by cut-off frequency they apply to a filter of any cut-off frequency. That is to say, the attenuation of a 1000 cycle low pass filter at 2 kc. and the attenuation of a 3 kc. filter at 6 kc. are both read from the same curve or table as 2·65 népers.

Similarly, the impedance curves have been plotted with the "nominal" (meaning the low frequency impedance $\sqrt{L/C}$) value as unit. Thus they apply to filters of any characteristic impedance just by a multiplication by 600 ohms (or whatever it is). Further, the curves apply equally to high pass filters, as will be shown later.

The Relation Between Cosh P and Attenuation

Cosh P needs to be over 1 in actual size to produce attenuation. If cosh P is negative and over 1 there is attenuation. This may be seen as ε^P is a current ratio and a phase reversal of current, i.e. a negative value to ε^P, say -6, makes ε^{-P} negative too, and so cosh P, which is

$$\frac{\varepsilon^P + \varepsilon^{-P}}{2},$$

is also negative. Hence, in looking for attenuation, cosh P, though it may be positive or negative, must be over 1, and a/b in ladder filters can have *either* any positive value *or* any negative value over 4, in order to make cosh P greater than 1.

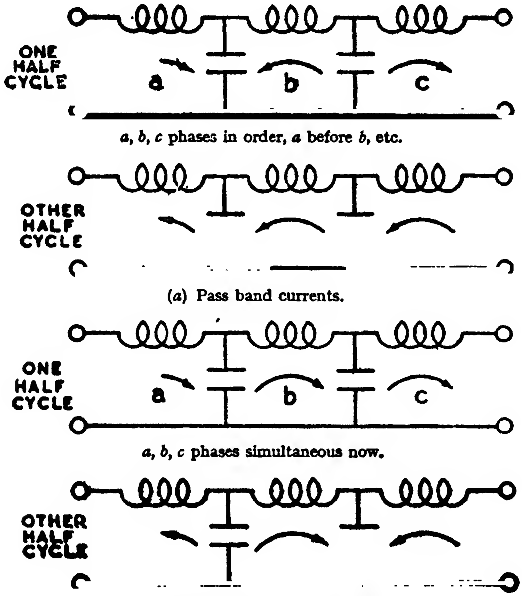
The one case is that of two similar impedances which act like a potentiometer and thus reduce the voltage, so cosh P is $+$ *ve*. The second is inductance for one arm and capacity for the other, so cosh P is $-$ *ve*. The sketch in Fig. 16 is not only theoretical, for any two terminal impedances may at any one frequency measure or "look" like a capacity and at another "look" like an inductance. For example, a coil and condenser in parallel look like a condenser at high frequencies, but like a coil at low frequencies. This is an easy way of following the working of difficult circuits.

In particular, the half-section shown in Fig. 16 has resonance between the inductance and capacity at the cut-off frequency.

When attenuating, the simple low pass filter has a phase change of 180° per section. This means that the phases are as shown in Fig. 24 (*a*).

The filtering action is roughly obvious since the series inductances oppose the passage of the higher frequencies, while a condenser allows them to pass.

Sometimes both arms of a ladder filter look like coils, in which case (Fig. 21) the filter obviously attenuates, as it is like a repeated potentiometer. A compound circuit changes as the frequency is



a, b, c phases in order, *a* before *b*, etc.

(*a*) Pass band currents.

a, b, c phases simultaneous now.

(*b*) Attenuation band currents.

FIG. 24. PHASE CHANGES IN LADDER FILTERS

raised through a resonance, so it may look like one thing at one frequency and another at a higher frequency. In the low pass filter it is plain that the condensers shunt the voltage across the line at high frequency.

Mathematics show that this happens at a critical frequency, and that there is no reduction below that frequency.

EXAMPLE

Design a low pass filter to have a nominal impedance of 600Ω and a cut-off of 1000 ~. It is required to have a high impedance in the *attenuation* band at one end and a low impedance at the other end. It should be balanced to earth.

$$L = \frac{R}{\pi f}, \text{ so } L = \frac{600}{1000\,\pi} = 0{\cdot}1909 \text{ henry}$$

$$C = \frac{1}{\pi R f}, \text{ so } C = \frac{1}{600{,}000\pi} = 0{\cdot}53\ \mu\text{F}$$

These are the *full* coils, and each coil needs to be wound with two windings for balance to earth. The 191 millihenrys are with *both* windings in series.

If double-shunt ended, begin with a

$$\frac{0{\cdot}53}{2} = 0{\cdot}265\mu F$$

condenser. If half-series ended, begin with a 95·5 mH coil.

Before going on to consider more complicated ladder circuits it will be well to look at the simple high pass filter.

The High Pass Filter

This has series condensers and shunt inductances. If $jL\omega$ is put for b in the general formulas and

$$\frac{1}{jC\omega}$$

for a, then the characteristic impedance and attenuation can be worked out; for, again,

$$Z_0 = \sqrt{ab + \frac{a^2}{4}}$$

in a ladder, and $a = \dfrac{1}{jC\omega}$, but $b = jL\omega$ so

$$Z_0 = \sqrt{\frac{L}{C} - \frac{1}{C^2 4\omega^2}} = \sqrt{\frac{L}{C}}\sqrt{1 - \frac{1}{LC4\omega^2}}.$$

$\therefore\ 4LC\omega^2 = 1$ for the cut-off frequency, defining the change of impedance from the real to reactive.

Also, at frequencies high now,

$$R = \sqrt{\frac{L}{C}}$$

$$\therefore\ LC = \frac{1}{4\omega^2} \text{ or } \sqrt{LC} = \frac{1}{2\omega} = \frac{1}{4\pi f_0}$$

So

$$L = \frac{R}{4\pi f_0} \text{ and } C = \frac{1}{4\pi R f_0}$$

This gives the sizes of components, but it is far easier to use the Theorem for Inversion about a given frequency (see Chapter IX). Take the main series impedance of the low pass filter, a coil of value

$$L = \frac{R}{\pi f_0} \text{ henrys}$$

The condenser which would resonate with this at f_0 is one which makes

$$\sqrt{\frac{C^1 R}{\pi f_0}} = \frac{1}{2\pi f_0}$$

or

$$\frac{C^1 R}{\pi f_0} = \frac{1}{4\pi^2 f_0^2} \text{ or } C^1 R = \frac{1}{4\pi f_0}$$

$$C^1 = \frac{1}{4\pi f_0 R}$$

which is the full condenser for a ladder high pass filter. By "full" condenser is meant the value to be placed in one side of the line when there is none in the other as in Fig. 25.

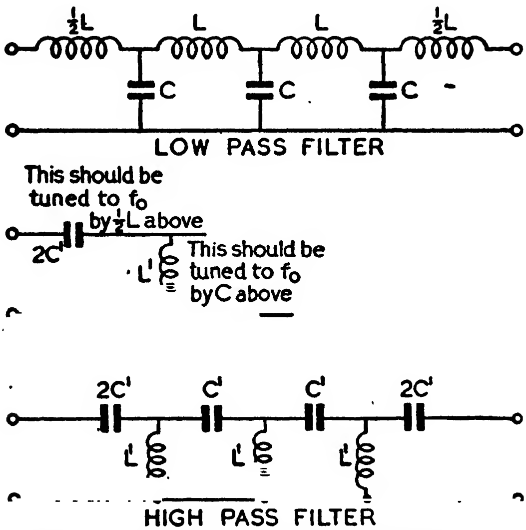


FIG. 25. DESIGN OF HIGH PASS FILTER BY FREQUENCY INVERSION OF LOW PASS FILTER

Applying the same principle to the condenser of the low pass filter gives a coil for the high pass filter of value L^1.

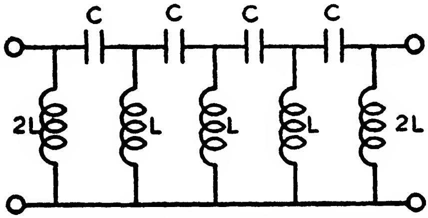
$$L^1 = \frac{R}{4\pi f_0} \text{ henry}$$

Compare the sizes of this coil and condenser with those for the low pass filter, and it will be seen that the full coil and condenser

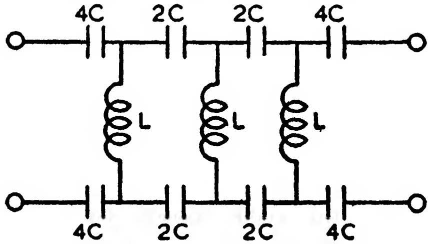
for a high pass ladder filter are *one-quarter the sizes* of the coil and condenser for a low pass filter of the same cut-off. So, knowing the low pass filter, one knows about high pass filters too.

The Two Terminations for Ladder High Pass Filters

If the high pass filter is to be "half-series" terminated, the method of getting half a series impedance is to double the end condenser. If it is to be double-shunt terminated then it must end not



(*a*) Double-shunt ended at each end. Unbalanced to earth.



(*b*) Half-series ended. Balanced to earth.

Fig. 26. The Ladder High Pass Filter

in a condenser but in a coil as shown in Fig. 26 (*a*), and the end coils must each be of inductance twice the value given by the formula

$$L = \frac{R}{4\pi f_0}$$

High Pass Filter Balanced to Earth

When it is desired to balance this filter to earth, every condenser must be doubled in size and also duplicated as in Fig. 26 (*b*), which shows such a filter half-series terminated. While this is at first a little confusing, it is all based on the fact that the impedance *a* in the general formulas is the loop impedance between two rungs of the ladder, two condensers of 2 μF in series acting like one of 1 μF, and also the fact that the general formulas are based on a value

of $\frac{1}{2}a$ or else $2b$ for the end impedance. It is the impedance which counts, and determines when to double, and when to halve, a coil or condenser to secure balance to earth and also make a certain termination. Table 7 gives examples of high pass filters.

TABLE 7

COMPONENTS FOR (NOMINAL) 600-OHM IMPEDANCE, HIGH PASS FILTERS

f	L Full Coil	C Full Condenser	$2L$ End Coil
	mH	μF	mH
62½	382	4·12	764
125	191	2·12	382
187	128	1·42	256
250	95·6	1·06	191
375	64	0·71	128
500	47·8	0·53	95·6
750	32	0·355	64
1000	24	0·265	47·8
1500	15·9	0·177	32
2000	12	0·132	24

The Impedance of the High Pass Filter

From the simple formulas for the characteristic impedance of the general ladder circuit

$$\sqrt{ab}\sqrt{1+\frac{a}{4b}}$$

for a half-series, and

$$\frac{\sqrt{ab}}{\sqrt{1+\frac{a}{4b}}}$$

for a double-shunt termination, the particular curves for the impedance of the high pass filter with either termination may be obtained. Note that $\sqrt{ab}$ or $\sqrt{\text{coil} \times \text{condenser}}$ (because a is a condenser now and b is a coil) becomes

$$\sqrt{jL\omega \times \frac{1}{jC\omega}} = \sqrt{\frac{L}{C}}$$

just as for a low pass filter. This is the nominal impedance. The factor

$$\sqrt{1+\frac{a}{4b}}$$

or its reciprocal in the double-shunt case multiplies the constant $\sqrt{\frac{L}{C}}$ and produces a curve.

We have worked out the low pass filter impedance curves. What are the high pass ones like?

In $\sqrt{ab}$ the frequency f or ω cancels. In $\frac{a}{4b}$ it does not cancel again, so $\sqrt{1+\frac{a}{4b}}$ contains ω. It is worked out thus—

a is $\frac{1}{jc\omega}$ and b is $L\omega$, so it is

$$\sqrt{1-\frac{1}{4LC\omega^2}}$$

Note the 4, but also that

$$L = \frac{R}{4\pi f_0} \text{ and } C = \frac{1}{4\pi R f_0}$$

for this filter, or

$$LC = \frac{1}{4\omega_0^2}$$

where ω_0 is the cut-off but ω any frequency.

This makes the impedance for the half-series case $\sqrt{1-x^2}$, but note that x here is not

$$\frac{\text{frequency}}{\text{cut-off frequency}} \text{ but } \frac{\text{cut-off frequency}}{\text{frequency}}$$

the reciprocal. Observing this means that the curves previously obtained for low pass filters are curves for high pass filters if one changes the scale at the base to reciprocal values. The simplest way is to use $\frac{1}{2}$ for 2, $\frac{1}{3}$ for 3, 2 for $\frac{1}{2}$, and so on.

As the frequency comes down in the high pass filter the result is just like going up in the low pass case. Compare the half-series ended low pass filter with the half-series ended high pass filter. The double-shunt high is like the double-shunt low pass filter curve.

How to Tell the Impedance Curve from the Circuit

There is a rapid way to see which curves to use for which case. The $\sqrt{1-x^2}$ for the series and

$$\frac{1}{\sqrt{1-x^2}}$$

for the shunt case are the same for both filters. One goes up as x

becomes very large, but is zero at cut-off, and the other goes down to 0 at infinity value of x, being infinite at cut-off.

From the circuit, does the filter end in a shunt condenser or a series coil? This quickly shows what it does at infinite frequency, and so gives the required information for that value of x.

Ladder Band Pass Filter

One of the most useful of band pass filters is the four-element one shown in Fig. 27. This drawing does not take into account balance to earth or terminations, which must be arranged after working out L_1, C_1, L_2, and C_2 from the formulas first of all.

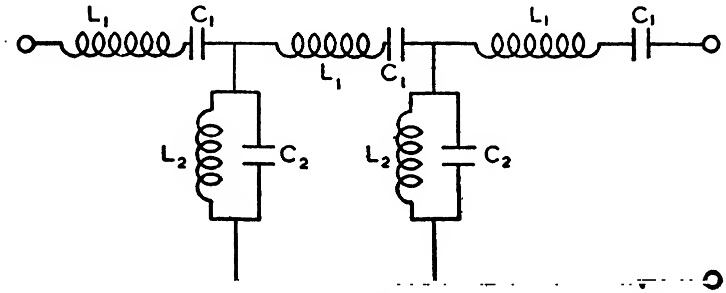


FIG. 27. SIMPLE LADDER BAND PASS FILTER (NOT TERMINATED)

As for the derivation of the formulas, suppose L_1 resonates with C_1 at the same frequency as that for which L_2 and C_2 are in anti-resonance. Then, neglecting losses, at this frequency L_1 and C_1 will together have zero impedance. Also L_2 and C_2 will be infinity and put no leak on the line. This frequency is, therefore, in the pass band. The filter is usually made to match the line round about this frequency as far as impedance goes.

To study the circuit, calculate its characteristic impedance from the general formula

$$Z_0 = \sqrt{ab}\sqrt{1 + \frac{a}{4b}}$$

Putting in the values for a and b, remembering that a is a series resonant and b a parallel resonant circuit,

$$Z_0 = \sqrt{\frac{\left(jL_1\omega + \frac{1}{jC_1\omega}\right)\frac{L_2}{C_2}}{jL_2\omega + \frac{1}{jC_2\omega}}}\sqrt{1 + \frac{\left(jL_1\omega + \frac{1}{jC_1\omega}\right)}{4\frac{L_2}{C_2}}\left(jL_2\omega + \frac{1}{jC_2\omega}\right)}$$

As
$$L_1\omega + j\frac{1}{jC_1\omega}$$
is to resonate at the same frequency as
$$jL_2\omega + \frac{1}{jC_2\omega},\ L_1C_1 = L_2C_2 \text{ and}$$
the first fraction reduces to
$$\sqrt{\frac{L_1}{L_2}\frac{L_2}{C_2}} = \sqrt{\frac{L_1}{C_2}}$$
Then when working at the frequency of resonance
$$Z_0 = \sqrt{\frac{L_1}{C_2}} = R$$
to match the line because the brackets
$$\left(jL_1\omega + \frac{1}{jC_1w}\right) \text{ and } \left(jL_2\omega + \frac{1}{jC_2\omega}\right)$$
are then both zero. The attenuation starts when Z_0 becomes unreal or when
$$\sqrt{1 + \frac{\left(jL_1\omega + \frac{1}{jC_1\omega}\right)\left(jL_2\omega + \frac{1}{jC_2\omega}\right)}{4\frac{L_2}{C_2}}}$$
becomes unreal, but this is when
$$\frac{\left(jL_1\omega + \frac{1}{jC_1\omega}\right)\left(jL_2\omega + \frac{1}{jC_2\omega}\right)}{4\frac{L_2}{C_2}} = -1$$
Notice that
$$\left(jL_2\omega + \frac{1}{jC_2\omega}\right) \text{ is } \frac{L_2}{L_1}\left(jL_1\omega + \frac{1}{jC_1\omega}\right)$$
because with the same resonance frequency one bracket is just a fraction of the other all the time. Use this and also take a square root so
$$\left(jL_1\omega + \frac{1}{jC_1\omega}\right)\left(\frac{L_2}{4L_1}\frac{C_2}{L_2}\right)^{\frac{1}{2}} = j$$
or
$$\frac{1}{2}\left(L_1\omega - \frac{1}{C_1\omega}\right)\sqrt{\frac{C_2}{L_1}} = 1$$

or $$L_1C_1\omega^2\sqrt{\frac{C_2}{L_1}} - \sqrt{\frac{C_2}{L_1}} = 2C_1\omega$$

or $$\omega^2 - \frac{2\omega}{\sqrt{L_1C_2}} - \frac{1}{L_1C_1} = 0$$

Call the frequencies α and β.

$$\alpha + \beta = \frac{2}{\sqrt{L_1C_2}} \text{ while } \alpha\beta \text{ is } + \frac{1}{L_1C_1}$$

Consider $$\left(L_1\omega - \frac{1}{C_1\omega}\right) = \pm\, 2\sqrt{\frac{L_1}{C_2}} = \pm\, 2R$$

When $\omega = \alpha$ the $-$ sign is taken, so

$$L_1\alpha - \frac{1}{C_1\alpha} = -\,2R$$

but when $\omega = \beta$ the $+$ sign is taken, so

$$L_1\beta - \frac{1}{C_1\beta} = 2R$$

Then $L_1C_1\alpha^2 + 2RC_1\alpha - 1 = 0$
and $L_1C_1\beta^2 - 2RC_1\beta - 1 = 0$

$$\left.\begin{aligned} \alpha &= \frac{-\,RC_1 + \sqrt{R^2C_1^2 + L_1C_1}}{L_1C_1} \\ \beta &= \frac{RC_1 + \sqrt{R^2C_1^2 + L_1C_1}}{L_1C_1} \end{aligned}\right\} \text{using the plus sign in both cases.}$$

So $$\alpha + \beta = \frac{2\sqrt{R^2C_1^2 + L_1C_1}}{L_1C_1}$$

Then

$$\alpha\beta = \frac{R^2C_1^2 + L_1C_1 - R^2C_1^2}{L_1^2C_1^2} = \frac{1}{L_1C_1}$$

and $$\beta - \alpha = \frac{2R}{L_1}$$

The four equations settling the sizes are now—

(1) . . . . $L_1C_1 = L_2C_2$

(2) . . . . $\dfrac{L_1}{C_2} = R^2$

(3) . . . . $\dfrac{1}{L_1C_1} = \alpha\beta$

(4) . . . . $\beta - \alpha = \dfrac{2R}{L_1}$

Calling the band edges frequencies of f_1 and f_2, $\alpha = 2\pi f_1$ while $\beta = 2\pi f_2$, we find the components as follows—

From (4) $L_1 = \dfrac{R}{\pi(f_2 - f_1)}$ $\dfrac{R}{\pi(f_2-f_1)} \times \dfrac{\pi R\,(f_2-f_1)}{} = R^2$

From (2) $C_2 = \dfrac{L_1}{R^2} = \dfrac{1}{\pi R\,(f_2 - f_1)}$

From (3) $C_1 = \dfrac{1}{\alpha\beta L_1} = \dfrac{\pi(f_2 - f_1)}{R2\pi f_1 2\pi f_2} = \dfrac{1}{4\pi R}\left\{\dfrac{1}{f_1} - \dfrac{1}{f_2}\right\}$

From (1) $L_2 = L_1\dfrac{C_1}{C_2} = \dfrac{R}{4\pi}\left\{\dfrac{1}{f_1} - \dfrac{1}{f_2}\right\}$ $R = 2 \quad R^2 = 4 = \dfrac{L_1}{C_2}$

The Components of the Band Pass Filter or $L_1 = 4C_2$

The following four expressions give the four components for the plain band pass filter.

Series coil $= \dfrac{R}{\pi(f_2 - f_1)}$ $= \dfrac{R}{\pi(10{,}000)}$

Shunt condenser $= \dfrac{1}{\pi R(f_2 - f_1)}$ $= \dfrac{1}{\pi R(10{,}000)}$

Series condenser $= \dfrac{1}{4\pi R}\left\{\dfrac{1}{f_1} - \dfrac{1}{f_2}\right\}$ $= \dfrac{1}{4\pi R}\left(\dfrac{1}{451} - \dfrac{1}{461}\right)$

Shunt coil $= \dfrac{R}{4\pi}\left\{\dfrac{1}{f_1} - \dfrac{1}{f_2}\right\}$ $= \dfrac{R}{4\pi}\left(\dfrac{1}{451} - \dfrac{1}{461}\right)$

These formulas are much more easily remembered if it is observed that L_1 is the inductance for a low pass filter whose cut-off frequency is the "band width" of the desired band pass filter.

The same remark applies to the condenser C_2. On the other hand, the shunt coil L_2 is the difference of the coils for two high pass filters, one of cut-off f_1, the other of cut-off f_2. The series condenser is also the difference of two condensers suitable for two high pass filters, one for each of the frequencies which form the edges of the band in question.

These results are summarized in a table later and enable ordinary ladder filters to be designed from memory and sometimes almost mentally. (See page 49.)

Example

As an example of the band pass filter, take the following. Design a band pass filter for a 1000Ω line to pass frequencies from 2000–5000 c/s. The width of the band is 3000 c/s. A low pass filter for 3000 c/s would need a coil

$$\frac{1000}{\pi 3000} = 0{\cdot}106 = 106\text{ mH}$$

This, then, is the full series coil. The shunt condenser is

$$\frac{1}{1000\ \pi 3000} = 0{\cdot}106\ \mu\text{F}$$

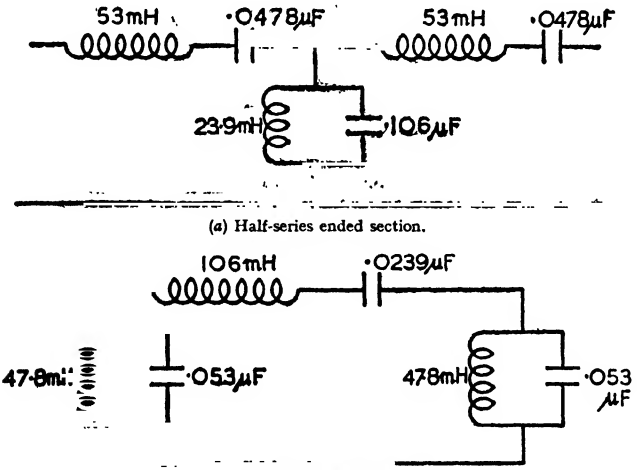
The series condenser is

$$\frac{1}{4000\ \pi 2000} - \frac{1}{4000\ \pi 5000}$$

Using the formulas for condensers for high pass filters,

$$C_1 = \left(\frac{1}{8\pi} - \frac{1}{20\pi}\right)\ \mu\text{F} = 0{\cdot}0239\ \mu\text{F}$$

The shunt coil is likewise 0·0239 henry = 23·9 mH. The fact that the figures for L_1 and C_1 are the same figures as those for C_2 and L_2 is a result of the impedance being 1000Ω. With an impedance of, say, 600Ω, this would not have happened.



(*a*) Half-series ended section.

(*b*) Double-shunt ended section.

Fig. 28. Numerical Examples of Band Pass Filter Sections

Terminations of Ladder Band Pass Filters

If a half-series termination is desired the end series coil must be half 106 mH, or 53 mH. The end condenser must have half the impedance of the 0·0239 condenser calculated above, so it must be 0·0478 μF.

If it be desired to use a double-shunt termination, the double shunt coil is 47·8 mH, and the condenser half 0·106 μF = 0·053 μF The two arrangements are shown in Fig. 28.

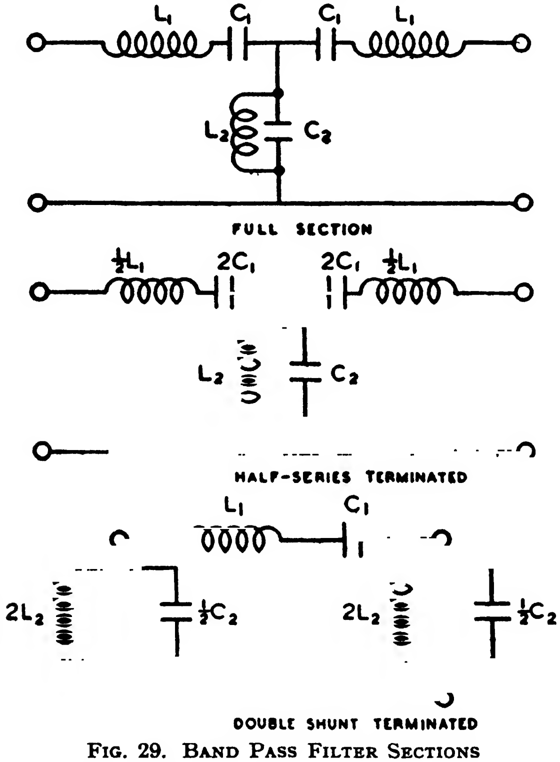


Fig. 29. Band Pass Filter Sections

Fig. 29 shows the full section as it would be in an infinitely long filter, and the two terminations as applied to a single section only.

It will be understood that in a filter of several sections it is only the end coil and condenser which has the half or double impedance values shown in Figs. 28 and 29. The interior of the filter has the ordinary values. In the pass band, the filter matches the line at the geometric mean frequency $\sqrt{f_1 f_2}$. In the above example this is 3160 c/s. At

other frequencies in the pass band, that is, as one or other of the band edges 2000 and 5000 c/s is approached, the half-series terminated filter falls in impedance to zero ohms at the cut-off, after which it rises in the attenuation band to infinity ohms for D.C. and also at an infinite number of cycles. This last is readily seen, because of the coil and condenser in series at the beginning of the filter. See also Table 8 for examples of a section of a band pass filter half-series terminated.

The double-shunt terminated filter has a rising impedance in the pass band as one or other of the cut-off frequencies is approached. It is infinite at the cut-off, and after changing to a reactance gradually falls to zero ohms at very low or very high frequencies. The half-series ended filter, like all the rest, is reactive as regards its characteristic impedance outside the pass band.

The question of the extra attenuation produced by these changes in impedance is dealt with later.

Simple Explanation of this Band Pass Filter in Action

The series arm L_1C_1 and the shunt arm L_2C_2 have the same resonance frequency.

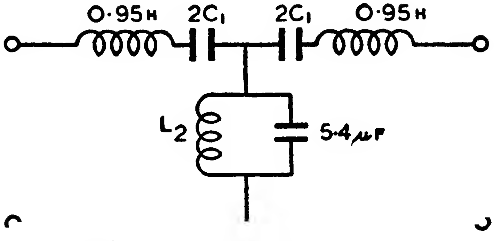
TABLE 8

SERIES CONDENSER AND SHUNT COIL FOR A BAND PASS FILTER 100 CYCLES WIDE, HAVING 600 OHMS IMPEDANCE

	Series Condenser	Shunt Coil	Value of Series Condenser (End Value)
f_1	C_1	L_2	$2C_1$
		mH	μF
80– 180	0·925	0·333	1·85
150–250	0·353	0·127	0·71
200–300	0·22	0·080	0·44
250– 350	0·1515	0·054	0·303
300– 400	0·108	0·039	0·217
350– 450	0·084	30·2	0·168
450– 550	0·0536	19·3	0·107
550– 650	0·037	13·3	0·074
650– 750	0·0272	10	0·054
750– 850	0·0208	7·5	0·042
850– 950	0·01645	5·9	0·033
950–1050	0·0133	4·8	0·027
1050–1150	0·01095	4	0·022
1450–1550	0·0059	2·2	0·012
2000–2100	0·00316	1·1	0·0063

The series coil full value is $\frac{600}{\pi \times 100}$, nearly two henries, and constant if the band width is constant, as it is (100 cycles) in this case. The very end coil to go with the end condenser is just under 1 henry to make a half-series ended filter.

The sketch shows the components for a single section of the example in Table 8.



(The half-series termination is shown)

The resonances occur in the pass band at the geometric mean of the two frequencies f_1 and f_2 of the band edges, as shown in Fig. 30 (*a*). The curves are steeper the narrower the band. Band width is the number of cycles $f_2 - f_1$ divided by $\sqrt{f_1 f_2}$.

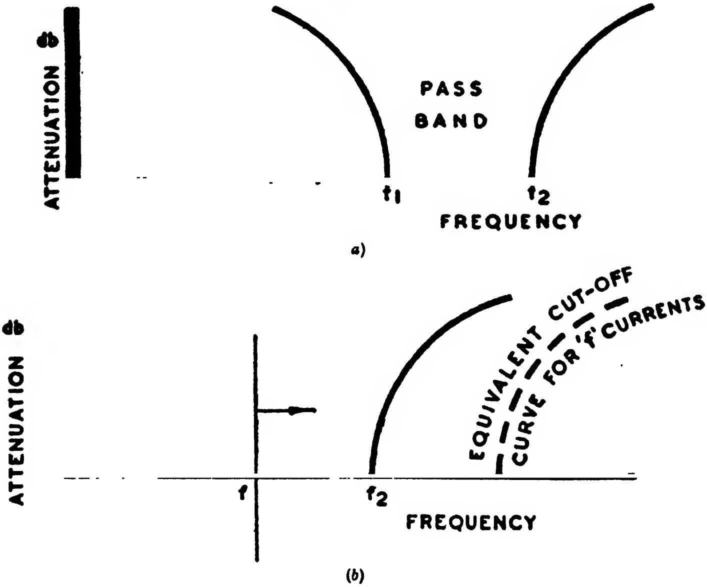


FIG. 30. ACTION OF BAND PASS FILTER TO CURRENTS IN THE PASS BAND

The result is that at a frequency a little *above* this, the series combination is an inductance, and the shunt circuit is a capacity,

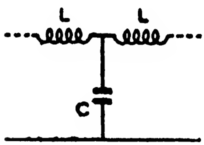
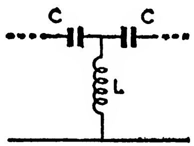
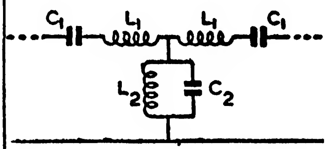
so that the filter acts like a low pass filter. The cut-off of this equivalent is very high, being somewhere over to the right of f_2 (see Fig. 30 (*b*)), but as higher frequencies are taken in the band $f_1 - f_2$ the value of the series inductance appears to increase and also the apparent shunt capacity, so the cut-off of the equivalent low pass filter is lowered until when the frequency f_2 is used the cut-off is actually f_2.

At frequencies above f_2 the filter acts like a low pass filter with a cut-off *below* f_2, and so there is a quick attenuation.

At low frequencies the affair acts like a high pass filter. The following table shows at a glance the sizes of inductances and capacities needed for simple low, high, and band pass filters.

TABLE 9

TABLE FOR CALCULATING VALUES OF INDUCTANCE AND CAPACITY OF LADDER FILTERS

Low Pass	High Pass	Band Pass
		
$L = \frac{R}{\pi f}$ $C = \frac{1}{\pi R f}$.	Use a quarter of values of L and C for a low pass filter	L_1 as for a low pass filter of same band width
		C_1 difference of two high pass filters for f_1 and f_2
		L_2 difference of two high pass filters for f_1 and f_2
		C_2 as for a low pass filter of band width

Balance to Earth of Band Pass Filters

To put half the series impedances in each line where balance to earth is wanted means doubling every condenser in the line and inserting this *double value in each line*.

The coils can be wound in two windings on the same core, and are thus no different for a balanced filter as regards material. In

balancing to earth in this way the shunt impedances are untouched. We must now deal with more complicated ladder circuits.

Derived Ladder Filters

The next type of ladder filter in order of complexity is called the series- or shunt-*derived* type. There are two varieties of the derived filter, and in both cases an extra component or two is used in each derived section of the filter.

The derived section has two advantages. It has a much sharper cut-off than the simple section, and also the impedance in the pass band can be made much more level with change of frequency. After the first steep rise in attenuation, however, the attenuation falls to a low value with further change of frequency. This is a disadvantage.

These sections are, therefore, good as end sections with one or more simple sections in the middle to keep up the attenuation at frequencies away from the cut-off. It pays often to use all derived sections with different peak frequencies.

The peak attenuation frequencies need to be correctly spaced out, however.

The result is to make an excellent filter with few components. If it is desired to save coils rather than condensers because of their higher cost then a shunt-derived section may or may not be better than a series-derived section.

The study of the derived filter can best be made by taking the low pass filter as an example.

The Derived Low Pass Ladder Filter

It will be noticed that the ordinary low pass filter has a very variable impedance in its pass band. This is common to all simple ladder filters. By putting a portion of the series impedance in the shunt arm, as shown in Fig. 31 and using smaller coils and condensers for the main ones, a low pass filter section is obtained which exactly matches the ordinary filter impedance when both are half-series terminated, but has a much better characteristic impedance when the derived section is double-shunt terminated. Filters may be shunt-derived, and the termination to be used to get the benefit from it as regards impedance is then half-series, as shown later.

The impedances of these filters, so terminated, are nearly level throughout the pass band. The series-derived one, double-shunt terminated, has a reciprocal impedance to the shunt-derived one, half-series terminated; when the one has fallen with a rising frequency to half its value, i.e. from the nominal value of 600 to 300

ohms, the other characteristic impedance would have risen from 600 to 1200 ohms, i.e. double.

The exact method of deriving these filters is as follows. Take the formula for characteristic impedance of the half-series terminated filter

$$Z_0 = \sqrt{ab + \frac{a^2}{4}}$$

Suppose now that the coil a is reduced to a fraction m of its simple value and a portion of inductance which we may call L_2 is put in series with b, which is a condenser in this case of a low pass filter. The characteristic impedance now is

$$\sqrt{ma\,(jL_2\omega + b) + \tfrac{1}{4}m^2a^2}$$

The question is, if this filter is to match the ordinary one in impedance, what change in the value of b, or what size of shunt arm coil L_2, will be needed? Call its impedance x. If b is increased in size to $\frac{b}{m}$ of its impedance (incidentally it will be a smaller condenser), then there is a portion in the formula $ma \times \frac{b}{m} = ab$ as before. The rest

$$\left\{max + \frac{m^2a^2}{4}\right\}$$

should be made equal to a^2. Then the new impedance exactly equals the old one. This gives an equation

$$max + \frac{m^2a^2}{4} = \frac{a^2}{4}$$

Solving for x gives

$$x = \frac{a^2(1-m)}{4am} = \frac{a(1-m^2)}{4m}$$

Hence the impedance, in this case a coil, is one of value $\frac{1-m^2}{4m}$ of the ordinary full series coil. As coil impedances are proportional to inductances, the fraction $\frac{1-m^2}{4m}$ is the fraction of the ordinary inductance. (Had it been a condenser its size would have been $\frac{4m}{1-m^2}$ to get the same impedance change. This is what happens in the high pass case.) The derived filter is shown in Fig. 31.

In increasing the impedance of the b arm notice that although it is $\frac{b}{m}$, a larger value of impedance, being a condenser this is obtained by using a smaller size condenser of value mC.

Let us sum up what has been done so far. Note that in Fig. 31 nothing has been said about terminations yet.

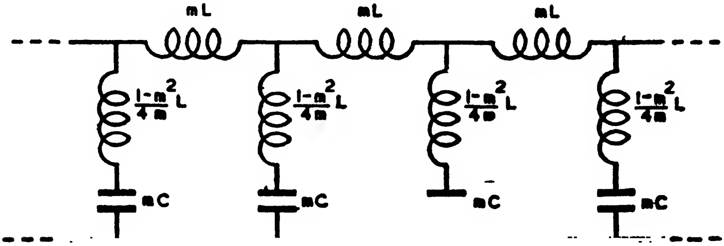


FIG. 31. SERIES-DERIVED LOW PASS LADDER FILTER (NOT TERMINATED)

It is the half-series terminated filter having this derived circuit whose impedance exactly matches the half-series ended simple filter at all frequencies in and out of the pass band, and when we say this we are speaking of characteristic impedances which are the impedances of infinite chains of similar sections.

The propagation constant of this derived filter has been altered, indeed the circuit owes its sharp peak of attenuation to the resonance of the shunt coil and condenser putting a short circuit on the line at some frequency above the cut-off, a circuit device which was pointed out before this filter was invented, but, to keep to impedances, consider what the impedance frequency curve of the filter becomes when it is not half-series terminated, but double-shunt terminated.

The answer depends on the value of m, but as the double-shunt terminated filter has a characteristic impedance

$$\frac{\sqrt{ab}}{\sqrt{1+\frac{a}{4b}}}$$

The simplest way to work this expression out is to multiply by $\sqrt{ab}$ top and bottom. It becomes

$$\frac{ab}{\sqrt{ab+\frac{a^2}{4}}}$$

Here a is now $jmL\omega$, while b is

$$\frac{1}{jCm\omega} + j\frac{1 - m^2}{4m} L\omega$$

The result is

$$\frac{ab}{\sqrt{ab + \frac{a^2}{4}}}$$

$$= \frac{jmL\omega\left(\frac{1}{jCm\omega} + j\frac{1 - m^2}{4m} L\omega\right)}{\text{(The usual mid-series impedance)}}$$

$$= \frac{\frac{L}{C} - L^2\omega^2\left(\frac{1 - m^2}{4m}\right)}{\sqrt{\frac{L}{C}(1 - x^2)}}$$

$$= \frac{\frac{L}{C}\left(1 - \frac{LC}{4}\omega^2(1 - m^2)\right)}{\sqrt{\frac{L}{C}(1 - x^2)}}$$

But $\frac{LC}{4} = \frac{1}{\omega_0^2}$ and $\sqrt{\frac{L}{C}} = R$ so, if the frequency f divided by the cut-off frequency f_0 is called x, then

$$\frac{\text{Impedance}}{R} = \frac{1 - x^2(1 - m^2)}{\sqrt{1 - x^2}}$$

for the *series-derived* filter *double-shunt ended* compared with

$$\sqrt{\frac{1}{1 - x^2}}$$

for the simple-shunt terminated filter. It is obvious that for the smaller values of x at any rate, that is, for the lower frequencies in the pass band, the effect of the changing denominator is neutralized somewhat by the numerator, the result being to keep the impedance more level in the pass band.

The method of terminating a simple ladder filter with a series-derived portion, then, is to suppose the simple filter half-series ended and put on a half-section, say, of the series-derived filter as

shown in Fig. 32. The flattest result obtainable in the pass band by varying the value of m is seen in Figs. 34 and 35, which show the impedance frequency curves for different values of m in the pass band

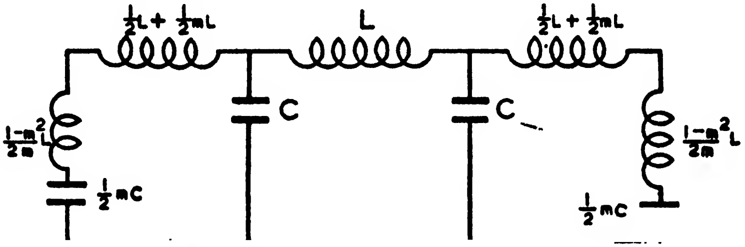


FIG. 32. A COMPOSITE FILTER WITH HALF A DERIVED SECTION AT EACH END

and the impedance and attenuation for a value of $m = 0{\cdot}7$ in the attenuation band. The full impedances in the attenuation band are contained in Tables 10 and 11, as well as the impedances in the pass band.

The $\frac{1}{2}$ (mL) of the half-section derived portion may be in one coil with the $\frac{1}{2}L$ of the simple ladder, giving a coil of $\left(\frac{1+m}{2}\right)L$ where L is the full coil for the simple filter.

The following tables show the impedance of filters in the pass and attenuation bands. The formula is that for the shunt-derived but half-series terminated ladder low pass filter and is the same as that for the Cauer Class I filter. (See also Fig. 40(b), page 64.)

TABLE 10

IMPEDANCE OF FILTERS IN THE PASS BAND

$$\frac{K}{R} = \frac{\sqrt{1-x^2}}{1-(1-m^2)x^2}$$

$m =$	0·3	0·4	0·5	0·6	0·7	0·8	0.9	1·0
$x =$								
0·1	1·005	1·003	1·003	1·001	1·0	0·99	0·997	0·995
0·2	1·014	1·01	1·008	1·005	1·0	0·99	0·985	0·979
0·3	1·042	1·03	1·022	1·011	0·99	0·99	0·97	0·954
0·4	1·071	1·057	1·04	1·02	0·99	0·972	0·943	0·916
0·5	1·116	1·096	1·065	1·031	0·99	0·951	0·908	0·866
0·6	1·189	1·148	1·094	1·04	0·98	0·92	0·856	0·8
0·7	1·29	1·21	1·13	1·04	0·954	0·868	0·784	0·7145
0·8	1·44	1·295	1·155	1·016	0·89	0·782	0·678	0·6
0·9	1·65	1·36	1·16	0·906	0·74	0·62	0·51	0·4352
0·95	1·73	1·28	0·96	0·736	0·578	0·463	0·368	0·3123

The curves are given in Fig. 34. The impedance after cut-off is shown in Fig. 35.

TABLE 11

IMPEDANCE OF FILTERS (LADDER-DERIVED) IN THE ATTENUATION BAND

$m=$	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
$x=$ 1·1	− 4·587	− 30·55	+ 4·96	+ 2·03	+ 1·2	+ 0·817	+ 0·595	+ 0·4587
1·2	− 2·145	− 3·17	− 8·3	+ 8·46	+ 2·5	+ 1·37	+ 0·914	+ 0·6641
1·3	− 1·542	− 1·473	− 3·11	− 10·01	+ 6·02	+ 2·12	+ 1·225	+ 0·8807
1·4	− 1·255	− 1·517	− 2·08	− 3·86	+ 2·450	+ 3·36	+ 1·56	+ 0·9798
1·5	− 1·065	− 1·258	− 1·63	− 2·54	− 7·6	+ 5·9	+ 1·95	+ 1·1188
1·6	− 0·934	− 1·078	− 1·34	− 1·945	− 4·05	+ 15·9	+ 2·43	+ 1·24
1·7	− 0·85	− 0·986	− 1·235	− 1·643	− 2·91	− 34·2	+ 3·01	+ 1·378
1·8	− 0·765	− 0·864	− 1·04	− 1·39	− 2·29	− 8·94	+ 3·88	+ 1·49
1·9	− 0.71	− 0·794	− 0·95	− 1·236	− 1·927	− 5·4	+ 5·16	+ 1·621
2·0	− 0·653	− 0·732	− 0·866	− 1·112	− 1·666	− 4	+ 7·22	+ 1·732

TABLE 12

SERIES-DERIVED LOW PASS LADDER FILTERS

Factor m	$\frac{\text{New Series } L}{\text{Old Series } L}$	$\frac{\text{New Shunt } C}{\text{Old Shunt } C}$	$\frac{\text{Shunt Coil}}{\text{Old Series } L}$
1·0	1	1	0
0·9	0·9	0·9	0·052
0·8	0·8	0·8	0·1125
0·7	0·7	0·7	0·182
0·6	0·6	0·6	0·27
0·5	0·5	0·5	0·375
0·4	0·4	0·4	0·52
0·3	0·3	0·3	0·67
0·2	0·2	0·2	1·2
0·1	0·1	0·1	2·5

These tables are intended as examples of the technique rather than as design information. The essence is always to work *in impedances* and *not* component sizes.

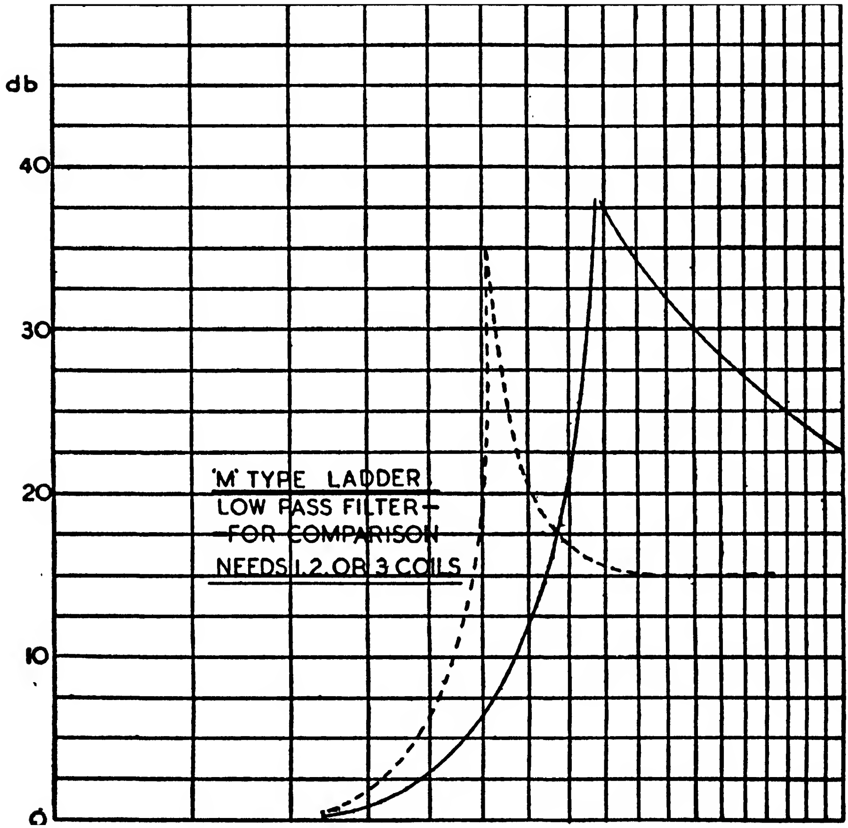


FIG. 33. TYPICAL ATTENUATION CURVE OF A FULL DERIVED SECTION

Two values of m are shown.

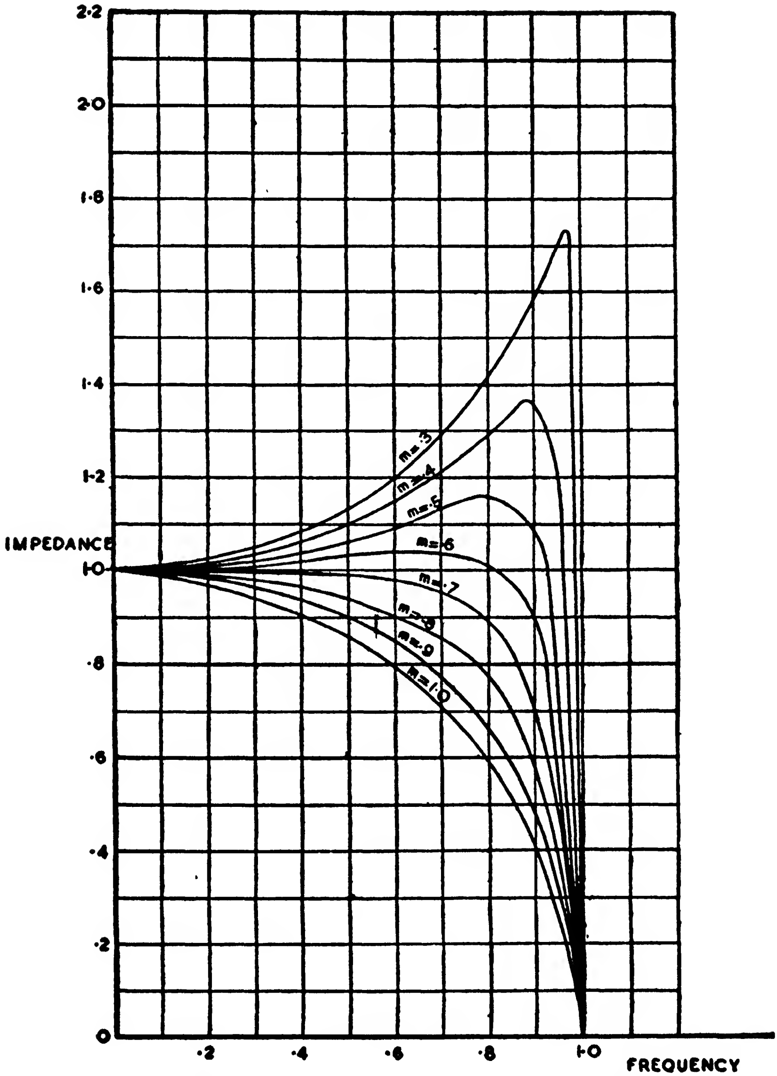


FIG. 34. IMPEDANCE CURVES OF DERIVED LADDER FILTERS IN THE PASS BAND

The curves show a shunt-derived filter with a half-series termination. The series-derived filter with double-shunt termination is the reciprocal of this.

Deriving High and Band Pass Filters

If a series-derived filter is desired, the rules are just the same as those for the low pass filter. They are—

(1) Choose the factor *m* from considerations of impedance in the

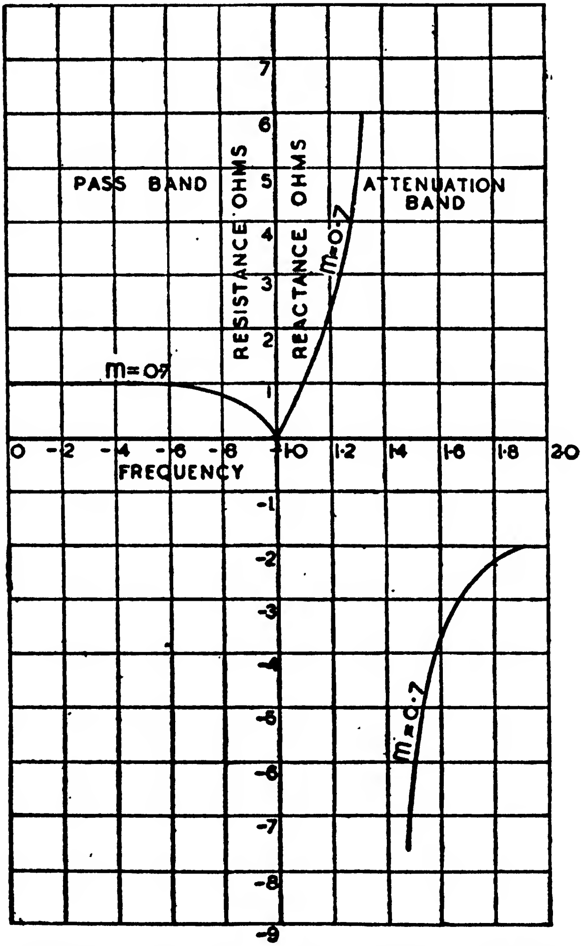


FIG. 35. IMPEDANCE CURVE OF A DERIVED LADDER FILTER SHUNT-DERIVED AND HALF-SERIES TERMINATED. VALUE OF $m = 0{\cdot}7$

(See Tables 10 and 11 for other values of *m*.)

pass band or else the attenuation which may be required to be a maximum at some particular frequency above the cut-off.

(2) Take m of the ordinary series impedance (i.e. less inductance and more capacity if it is a low pass filter).

(3) Take $\frac{1}{m}$ of the ordinary shunt impedance (i.e. more inductance or less capacity in a low pass filter).

(4) Form $\frac{1-m^2}{4m}$ of the ordinary series impedance calculated from Table 12.

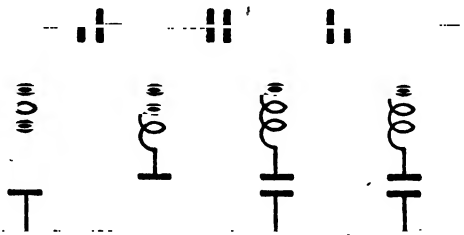


FIG. 36. SERIES-DERIVED HIGH PASS FILTER

(5) Connect these up in the proper way, with (3) and (4) in series to form a shunt arm.

(6) Take half of (2) to fit on to a simple series-terminated ladder filter.

(7) Take twice (5) to form the double shunt to go on the outside whenever a level impedance characteristic looking into the filter is desired.

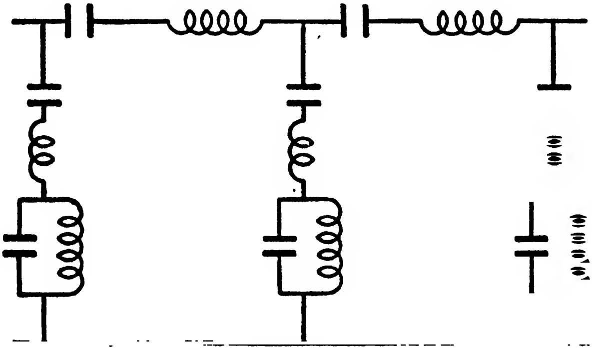
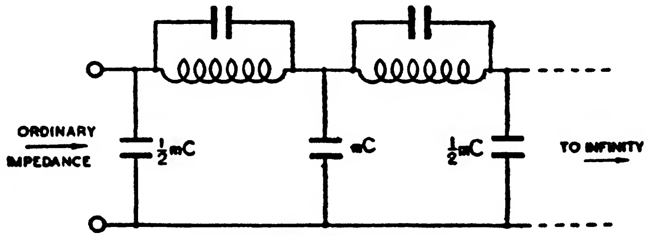


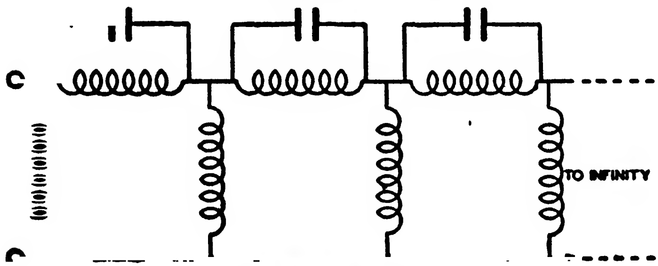
FIG. 37. SERIES-DERIVED BAND PASS FILTER

This and the previous example are terminated in such a manner as to give a derived filter a good impedance curve.

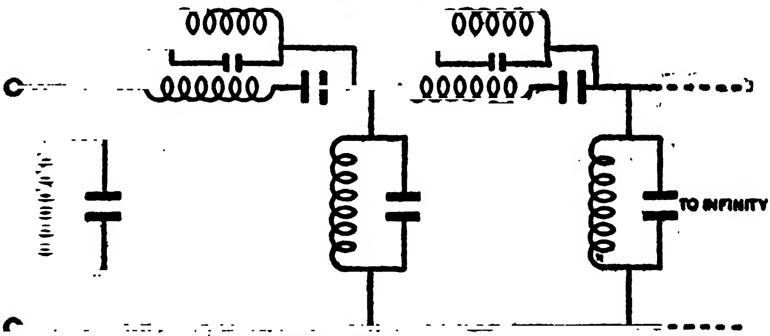
The series-derived high pass filter is shown in Fig. 36, and the series-derived band pass filter in Fig. 37 without regard to terminations. After terminating the filter as desired, balance to earth can be secured if it is necessary, by halving all series impedances and duplicating them one in each side of the line.



(a) Low pass filter, shunt-derived.



(b) High pass filter, shunt-derived.



(c) Band pass filter, shunt-derived.

FIG. 38. SHUNT-DERIVED FILTERS TERMINATED TO HAVE JUST THE SAME IMPEDANCE CURVE AS THE SIMPLE (UNDERIVED) FILTER

Shunt-derived Filters

The process here is to take the infinite filter not terminated at

all, that is, to use the full section component sizes of coils and condensers, and then place a piece of the shunt arm (a coil if it is a coil, a condenser if a condenser, and so on) in parallel with the series arm as in Fig. 38. The series arm is first reduced in its impedance by a factor m, and then the extra impedance put in parallel. The impedance of the shunt arm is raised in the proportion $1:m$. The result is an alteration in the attenuation curve.

It cannot be too strongly emphasized that when it is desired to give the filter a better *impedance frequency* curve the shunt-derived

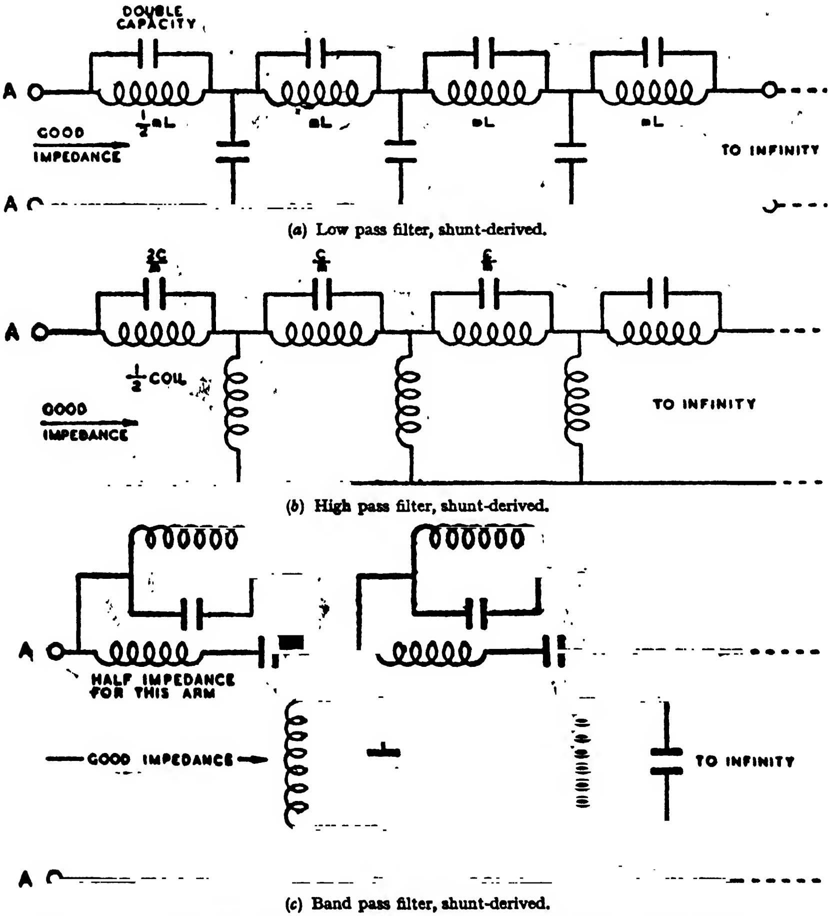


FIG. 39. SHUNT-DERIVED FILTERS TERMINATED FOR GOOD IMPEDANCE AT *AA*

filter, after derivation, must be half-*series* terminated, for if it is double-shunt terminated, as in Fig. 38 (*a*), its impedance is no different from that of the simple filter, though it has peaks of attenuation if it is a band pass filter and one peak if it is a high or low pass. Indeed, in getting out the formula to determine what fraction of the impedance of the plain-shunt arm ought to be taken and placed in parallel with the full plain series arm to give the desired circuit, the impedance of the derived filter, double-shunt terminated, is equated to that for the plain filter double-shunt terminated. The idea is to make it possible to join a double-shunt terminated derived filter to a double-shunt terminated plain filter without reflection loss. It is when terminated the other way, namely, half-series terminated after being shunt-derived, that a good impedance curve is obtained (see Fig. 39).

The rule is—

"DERIVE FIRST, THEN TERMINATE"

The following table shows the impedances and how they are obtained.

TABLE 13

IMPEDANCES OF FILTERS IN THE PASS BAND

	Derivation	Termination	Characteristics
1	Plain (not derived)	Half-series	Diagram like a quarter-circle *Poor:* falling to zero at cut-off
2		Double shunt	The reciprocal of the above *Poor:* rising to infinity at cut-off
3	Series-derived	Half-series	*Poor:* like (1)
4		Double shunt	Reciprocal of the one below *Good:* a level characteristic going to zero at cut-off
5	Shunt-derived	Half-series	The reciprocal of the above *Good:* a level characteristic rising to infinity at cut-off
6		Double shunt	*Poor:* like (2)

The Method of Derivation and Termination as it Affects Impedance in the Pass Band

The impedances of filters in the foregoing table are a little difficult to grasp at first. They apply to infinitely long filters, though one

may at any frequency break off a filter anywhere, even after the first half-section, and terminate it in its proper characteristic impedance, in which case it will measure like an infinitely long filter. The impedance formulas all refer to infinitely long filters.

Impedances of Filters in the Attenuation Band

The most useful thing is to know that the simple half-series ended filter, which falls to zero at the cut-off and changes to a reactance, rises towards infinity after cut-off. The formula for a low pass filter $Z_0 = R\sqrt{1 - x^2}$ in the pass band becomes $Z_0 = j\sqrt{x^2 - 1}$ in the attenuation band, a curve shown in Fig. 12.

If it is a double-shunt terminated filter, again not of the derived type, it is

$$Z_0 = \frac{-jR}{\sqrt{x^2 - 1}}$$

which is now a negative reactance falling towards 0 ohms after cut-off.

In this work the filter is first designed as a simple filter infinitely long, then derived, also infinitely long. Bits of these two are now selected and put together. If half a derived section is put at the end of, say, two simple sections, the half-section must be put so that the good impedance faces the apparatus to which the filter is connected. This is indicated in Fig. 40.

It is not essential to use derived sections at all. The derived section has one disadvantage, as already mentioned. Although it has a second advantage in that there is a particularly rapid rise in attenuation after the cut-off, this is offset somewhat by the fact that the attenuation, after its high peak value, falls to a small value. It does not keep high like the simple section.

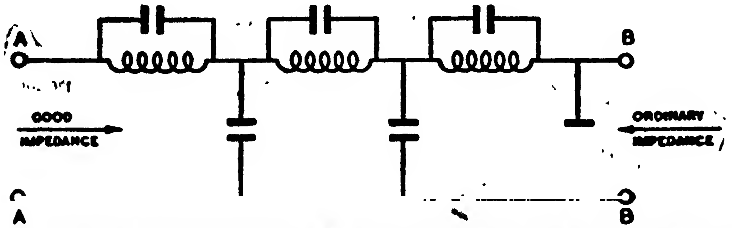
A whole group of curves is available in each case by varying the value of the parameter m. The earlier the peak the lower the value to which the curve falls afterwards.

Reciprocal Impedances

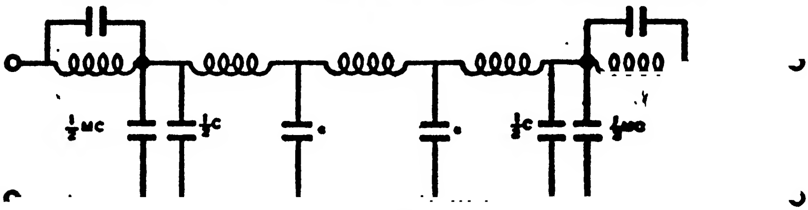
It can be seen from Theorem XI (Chapter IX) that the series- and shunt-derived filters, both derived from the same plain ladder, are inverse impedances. That is to say, their characteristic impedances multiply to produce R^2 at any and therefore at every frequency. The easiest way of making a *shunt*-derived filter is to make a *series*-derived filter and use Theorem XI, or else this theorem can be used to find general formulas.

All is contained in the slogan "*Derive first, then terminate.*"

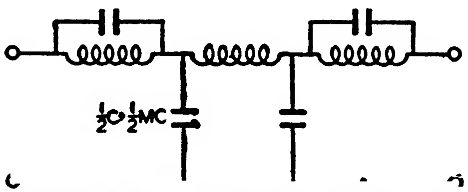
The following drawing, Fig. 40, shows the way in which a shunt-derived filter may have a good level impedance at one end and be no better than an ordinary plain filter at the other end. It can, even if only half a section, be joined to an ordinary filter as in Fig. 40 (*b*). Fig. 40 (*c*) shows the practice.



(*a*) The good impedance is found at *AA* when *BB* is connected to ordinary filter sections. The poor or ordinary impedance which is found at *BB* is useful, as it is suitable for connection to ordinary sections thus—



(*b*)



(*c*) Completed three-section filter with good impedance at both ends.

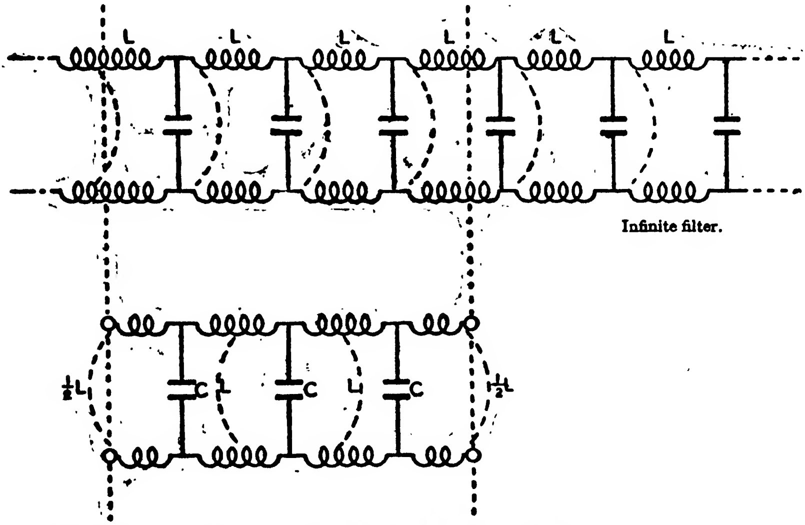
FIG. 40. THE USE OF DERIVED SECTIONS OR HALF-SECTIONS AT EACH END OF A FILTER TO GIVE GOOD IMPEDANCE TO MATCH THE TERMINAL APPARATUS

Two sections of different *m* values may be joined directly together, but each section must be half-series or else double-shunt terminated before they are put together. The low pass filter will then have two small shunt condensers together (if it is a shunt-derived filter), and in practice these are replaced by one large condenser.

Similarly in a series-derived band pass filter. Two *m* sections to be joined will each have half a series arm, each with coil and condenser. In series these form one coil and one condenser.

Building up Filters from Sections

When taking one or more sections to build into a complete filter the sections must be half-series or double-shunt terminated as shown in Fig. 41. These are ready then to go in a circuit or to have *m* portions put at the ends.



Portion taken out of the infinite filter showing half-series termination.

FIG. 41. FILTER SECTIONS IN USE

CHAPTER III

LATTICE FILTERS

SOME time after the invention of the ladder filter, a "phase corrector" circuit of lattice form was made. This contained series coils and "latticed" or crossed condensers. It passed all frequencies and gave phase change, but no attenuation. It was not obvious that other circuits would act as filters, i.e. have pass and attenuation bands. It was soon found, however, that filters could be of lattice rather than ladder form.

Any filter whose basic formulas are easy lends itself to better mathematical treatment.

The lattice filter has very simple formulas for its characteristic impedance and for its propagation constant. They are—

$$Z_0 = \sqrt{ab} \text{ and } \tanh \tfrac{1}{2}P = \sqrt{\frac{a}{b}}$$

The a and b are the impedances of the arms of the lattice as shown in Fig. 42.

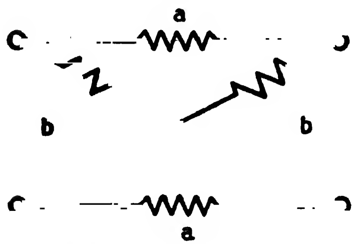


FIG. 42. LATTICE NETWORK

These two formulas come from using the "open" and "closed" method common in cable testing, which depends on reflection phenomena as shown by Oliver Heaviside.

From the circuit the open impedance is $\frac{1}{2}(a + b)$ as there are two parallel paths each with a and b in series. This is $Z_0 \coth P$. The closed impedance is seen to be twice a and b in parallel, which is $Z_0 \tanh P$, so we have

$$Z_0 \coth P = \tfrac{1}{2}(a + b)$$

$$Z_0 \tanh P = \frac{2ab}{a + b}$$

Multiplying gives $Z_0^2 = ab$, and division gives

$$\tanh^2 P = \frac{4ab}{(a + b)^2}$$

which is

$$\tanh P = \frac{2\sqrt{ab}}{a+b} = \frac{2\sqrt{\dfrac{a}{b}}}{1+\dfrac{a}{b}}$$

(The last step is a division by b in both top and bottom.)

It is plain that $\frac{P}{2}$ is as useful to us as P is.

Since $$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$$

this reduces to

$$\tanh \tfrac{1}{2}P = \sqrt{\frac{a}{b}}$$

which is the simple working formula for the attenuation of the lattice filter. The simplest lattice with reactance arms is a phase shift network, but we must first study a filter circuit.

If now the a arm is a coil making $a = jL\omega$ and the b arm a coil and condenser in series making

$$b = jL\omega + \frac{1}{jC\omega}$$

then the lattice acts like a low pass filter, as may be seen from the formula $Z_0^2 = ab$, which becomes

$$Z_0^2 = jL\omega\left(jL\omega + \frac{1}{jC\omega}\right)$$

in this case. The formula shows that it is a low pass filter, because when ω is below the resonance of L and C, then

$$\left(jL\omega + \frac{1}{jC\omega}\right)$$

is a negative reactance, making with the $jL\omega$ outside the bracket $j \times (-j)$ or $Z_0^2 = (+^{ve}$ real), which is a *pass* band. At higher frequencies, namely above resonance, it is $Z_0^2 = j \times j$, which is a *negative* real or an *attenuation* band.

The filter acts like a half-series terminated simple low pass ladder section. The coil L_1 in the a arm being made equal to the coil in the b arm makes $a = b$ at an infinite frequency, giving infinite attenuation there, which is correct. (See Fig. 43.)

It is possible by a mere change of coil size in the a arm, making b the bigger coil, to make a filter which will have the attenuation

characteristics of a series-derived *m*-type ladder low pass filter also half-series terminated. This also is shown in Fig. 43, and it will be seen that at an infinite frequency, when b becomes $jL\omega$, the b arm does not equal the $jL_1\omega$ of the a arm, so there is not an infinite attenuation there; but at a certain frequency above the cut-off, which makes the b arm look like an inductance equal to the coil in the a arm, there is an infinite attenuation. This is correct for an *m*-type ladder.

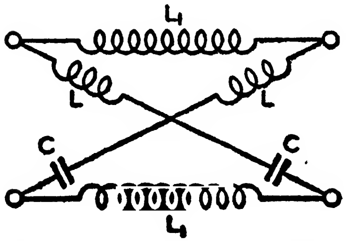


FIG. 43. $L_1 = L$ GIVES A SIMPLE SECTION. L LESS THAN L_1 GIVES A "DERIVED" SECTION

The attenuation of the lattice, depending as it does on tanh x, may be calculated from the graph of tanh x in Fig. 44. The x is in népers, and as the formula gives $\frac{P}{2}$, the value of x from the graph must be doubled to give P in népers. Then a factor gives decibels, for one néper is 8·686 db because log ε to base 10 is 0·4343 and db is 20 log x.

Reverting to Fig. 36, it will be remembered that in its invention the *m*-type ladder filter has the same impedance as the simple ladder

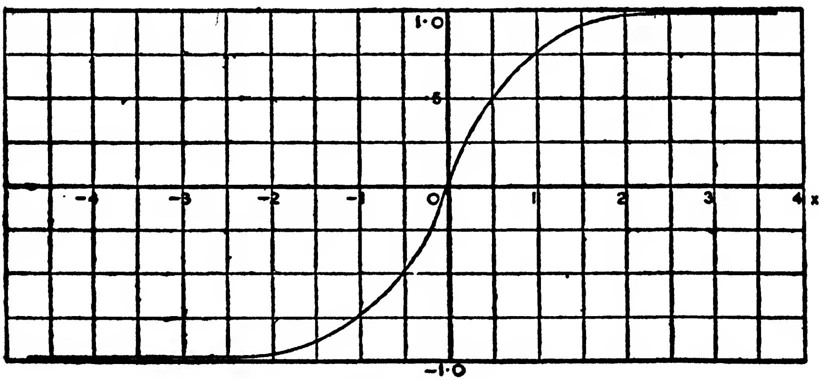


FIG. 44. TANH x

section if it is *half-series* terminated after being *series-derived*; so the lattice equivalent has the same poor impedance characteristic in the pass band as the simple ladder filter, namely a quadrant of a

circle beginning with a value $\sqrt{\frac{L}{C}}$ at low frequencies and falling to 0 at the cut-off, unless more complicated arms are used as Cauer does.

Equivalent Lattice and Ladder Formulas

It is possible to see what is the relation between a ladder and a lattice circuit for these two have exactly the same properties if we compare the cosh P formula of the ladder with the tanh $\frac{1}{2}P$ formula of the lattice. For the ladder

$$\cosh P = 1 + \frac{a}{2b}$$

but

$$\cosh P = 2\left(\cosh^2 \frac{P}{2}\right) - 1$$

in hyperbolic trigonometry, so

$$1 + \cosh P = 2 \cosh^2 \frac{P}{2} = 2 + \frac{a}{2b}$$

or

$$\cosh^2 \frac{P}{2} = 1 + \frac{a}{4b}$$

For this, using

$$\cosh^2 \frac{P}{2} - \sinh^2 \frac{P}{2} = 1, \text{ we have } \sinh^2 \frac{P}{2} = \frac{a}{4b}$$

so

$$\tanh \frac{P}{2} = \frac{\sinh \frac{P}{2}}{\cosh \frac{P}{2}} = \frac{\frac{1}{2}\sqrt{\frac{a}{b}}}{\sqrt{1 + \frac{a}{4b}}}$$

$$= \frac{\sqrt{a}}{\sqrt{a + 4b}}$$

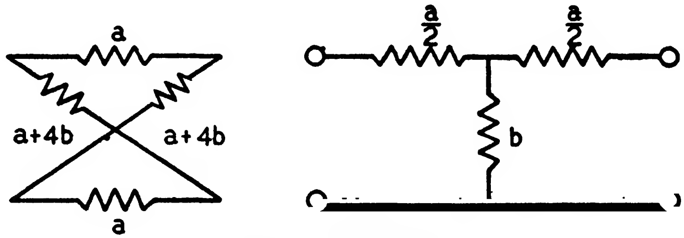
This is for a ladder. If now a lattice were made up with one arm the same as the a in the ladder and the other arm equal to $(a + 4b)$, i.e. a with four times b in series, then the new lattice would have the same attenuation as the ladder.

The characteristic impedance of the lattice would, however, be $\sqrt{a\,(a + 4b)}$, which is twice the impedance of the ladder as it is

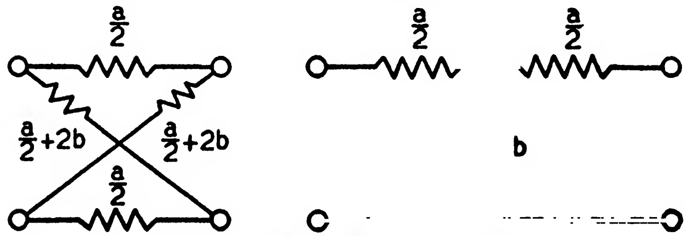
$$2\sqrt{ab + \frac{a^2}{4}}$$

If now the lattice has arms not a and $a + 4b$, but $\frac{1}{2}a$ and $\frac{1}{2}(a + 4b)$, then it will have the attenuation *and* the impedance of the ladder, as shown in Fig. 45.

Since, however, the lattice requires the b arm to contain the elements of *both* arms of the ladder, it does not seem to be as cheap



(a) Equal in attenuation only.



(b) Equal in impedance also and fully equivalent.

FIG. 45. LADDER AND LATTICE EQUIVALENTS

as the ladder. There is, however, something else to be said, and it is important. There may be lattices, not derived from ladders, which may be excellent filters and very economical, as in Fig. 46, but which have no ladder equivalent.

Further, where the lattice calls for a coil in each of the a arms, one coil will do the work of two if wound with twice the inductance (not twice the turns).

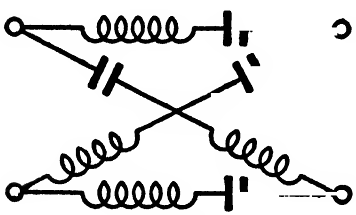


FIG. 46

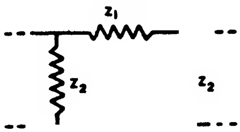
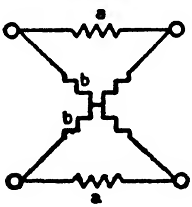
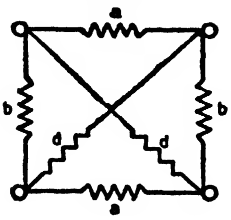
This does not apply to condensers, and so in itself does not go so far as another circuit change. There is a circuit equivalent of the lattice, which, using a transformer only, needs one a arm and one b arm, not two of each. This is a great saving in components.

With the equivalence of Fig. 45 one may make a lattice of any ladder, and so get a lattice with a good impedance characteristic.

It will be appreciated that other networks than the lattice will make a filter if suitable coils are put in, together with the right condensers. One may take a circuit and, by finding the "open" and "closed" impedance in terms of the separate arms with the far end first open then closed, derive general formulas for the characteristic impedance and for the attenuation constant in the form tanh $P =$ (an expression for the arms).

Once the formulas are derived using a, b, etc., as pure resistances for the arms, one may put any arrangement of coils and condensers in each arm and write the j notation terms in the formula in place of the pure resistances. Take, for example, the gate network in the following table of general formulas.

TABLE 14
GENERAL FORMULAS

Circuit	Impedance	Propagation Constant
Ladder 	Half-series ended: $Z_0 = \sqrt{Z_1 Z_2}\sqrt{1 + \frac{Z_1}{4Z_2}}$ Double-shunt ended: $Z_0 = \frac{\sqrt{Z_1 Z_2}}{\sqrt{1 + \frac{Z_1}{4Z_2}}}$	Cosh $P = 1 + \frac{Z_1}{2Z_2}$
Lattice 	$Z_0 = \sqrt{ab}$	Tanh $\frac{1}{2}P = \sqrt{\frac{a}{b}}$
Gate 	$Z_0 = \sqrt{(a \text{ and } b \text{ in parallel}) \text{ times } (d \text{ and } b \text{ in parallel})}$ which is $Z_0 = \sqrt{\frac{ab \times db}{(a+b)(b+d)}}$	Tanh $P/_2 = \sqrt{\frac{a \text{ and } b \text{ in par.}}{d \text{ and } b \text{ in par.}}}$

If a coil is put in for *a*, a condenser for *b*, and a coil and condenser in series for *d*, the gate filter becomes a low-and-band-pass filter. If now the sizes of coils and condensers are chosen so that the band has no width or else the attenuation band between the low pass and band pass region is zero, then a good low pass filter results and there are two alternative sets of sizes for the components, depending on which method of making the low-and-band-pass into a low pass filter was used.

The network, however, can be replaced by a lattice. Then the complicated lattice can be replaced by a simplification for a lattice network which saves components. This strongly suggests that the way to tackle filter design is to study many different networks. Without, however, going any further into filter design, one may employ the ladder formulas to build excellent filters, making use of the *m*-derived networks, and concentrating on those which give the best saving of coils if one is using expensive dust core coils.*

The Criterion for a Pass Band in Lattice Filters

If the arms of the lattice *a* and *b* are of the same sign in the *j* notation there is attenuation. If they differ in sign, that shows a pass band.

Since a complicated arm may change sign more than once as the frequency is raised from 0 to infinity there may be more than one pass band, but as practical circuits usually require one pass band and not more, those circuits treated in textbooks are arranged to have one pass band only. This is done by making the intermediate attenuation bands of zero width, i.e. knocking the pass bands into one big one, or by making some of the pass bands of zero width, or by a combination of the two. The formulas conceal this.

This is a suitable juncture for considering two matters that arise in practice. One is the effect of resistance losses, and the other is the effect of the reflections at the end of the filter when it is put into service.

* Dust core coils are now used almost always, except when intermodulation on a carrier system calls for air cores.

CHAPTER IV

LOSSES IN COMPONENTS

Effect of Losses in Filters

WHERE it is desired to take into account losses in coils and condensers in filter circuits, the impedance of a coil is no longer $jL\omega$ but $r + jL\omega$. This merely complicates the formulas for propagation constant and characteristic impedance. Take the "good impedance" gate low pass filter for which

$$\operatorname{Tanh}^2\frac{P}{2} = \frac{\dfrac{jC_2\omega}{j\omega L_2}\left(jL_2\omega - \dfrac{j}{C_2\omega}\right)\left(\dfrac{L_1}{C_2} - \dfrac{1}{C_1C_2\omega^2}\right)}{jL_1\omega - \dfrac{j}{C_1\omega} - \dfrac{j}{C_2\omega}}$$

as an example of a formula for P now complicated by resistance in the coil. Suppose now the coil L_1 has a resistance r_1, and L_2 has a resistance r_2; and it must be remembered that, to be exact, these r_1 and r_2 should be measured at every frequency for which the final result is wanted. As a rule, losses in condensers can be ignored if they are good mica ones. Then

$$\operatorname{Tanh}^2\frac{P}{2} = \frac{\dfrac{jC_2\omega}{r_2 + jL_2\omega}\left(jL_2\omega + r_2 - \dfrac{j}{C_2\omega}\right)\left(\dfrac{jL_1\omega + r_1}{jC_1\omega} - \dfrac{1}{C_1C_2\omega^2}\right)}{\left(jL_1\omega + r_1 - \dfrac{j}{C_1\omega} - \dfrac{j}{C_2\omega}\right)}$$

For example, at 800 c/s. let

$$L_1 = 0{\cdot}1908\,\text{mH} + 7^{\Omega}$$

$$L_2 = 95{\cdot}4\,\text{mH} + 3{\cdot}5^{\Omega}$$

$$C_1 = C_2 = 0{\cdot}133\mu\text{F}$$

These complex components are now put in the above formula and the result worked out as follows.

C_1 and C_2 have an impedance $-$ 1505 j, while the coil L_1 is 7 + 954 j and L_2 is 3·5 + 477 j.

$$\text{Tanh}^2 \frac{P}{2} = \frac{1\,(3\cdot5 + 477\,j - 1505\,j)\,(7 + 954\,j - 1505\,j) - 1505^2}{(-\,1505\,j)\,(3\cdot5 + 477\,j)\,(7 + 954\,j - 3010\,j)}$$

$$= \frac{(7 - 551\,j)}{(3\cdot5 + 477\,j)2} = \frac{511\,\sqrt{89^\circ 17'}}{2 + 477\,\underline{/89^\circ 38'}}$$

$$\text{Tanh}^2 \frac{P}{2} = 0\cdot578\sqrt{178^\circ 55'}$$

$$\text{Tanh}\,\frac{P}{2} = 0\cdot76\sqrt{89^\circ\,27'}$$

Here $\frac{P}{2}$ is evidently nearly all phase change $\frac{B}{2}$, which is right for a pass band. $\frac{B}{2} = \tan^{-1} 0\cdot76 = 37\frac{1}{4}^\circ$.

Actually it is $180^\circ - 37\frac{1}{4}^\circ$ because of the minus sign.

$$\text{Tanh}\,\frac{P}{2} = \frac{\tanh\frac{A}{2} + j\tan\frac{B}{2}}{1 + j\tanh\frac{A}{2}\tan\frac{B}{2}}$$

Because $\tanh\frac{B}{2}$ is nearly 90°, i.e. unreal, in the above, then (as $\frac{A}{2}$ is small) the value of $\tan\frac{A}{2}$ is the real part of $\tanh\frac{P}{2}$, i.e. $0\cdot76\sqrt{89^\circ\,27'}$ nearly.

Or $$0\cdot76 + \frac{33}{60} \times \frac{1}{57\frac{2}{11}} = 0\cdot006 \text{ néper}$$

making $$P = 0\cdot012 \text{ néper or } 0\cdot1 \text{ db}$$

The angle part is $180^\circ - 74\frac{1}{2}^\circ = 105\frac{1}{2}^\circ$.

The effect of losses in the filter is therefore very small if the coils are good ones with low losses, i.e. small resistance. It has only produced one-tenth of a decibel in this case.

Dealing with Vectors Nearly 0° or 90° or 180°

Suppose a quantity is nearly on one of the four quarter lines of the circle such as $0\cdot48\,\underline{/89^\circ}$ and it is desired to add another vector to it. The deviation of $\underline{/89^\circ}$ from $\underline{/90^\circ}$ may represent the losses in the circuit, and when the $0\cdot48\,\underline{/89^\circ}$ is turned to $(a + jb)$ it is important to preserve a, in this case the "real" portion of $a + jb$.

The unreal portion is 0·48, as the 1° deviation does not affect the length more than one part in 7000. The real portion is 0·48 sin 1°. Here it is well to remember that sin (small angle) = (angle in radians) and one radian = $57\frac{2}{11}°$. Then sin 1° = $\frac{1}{57\frac{2}{11}}$ = approx. 0·017, so $0{\cdot}48 \times 0{\cdot}017 = 0{\cdot}00816$. The vector is now $0{\cdot}00816 + 0{\cdot}48j$.

This is of value in dealing with circuits where losses in coils affect the overall characteristics of the circuit, and where it is desired to find how and to what extent the losses do affect matters.

In filter circuits the losses only affect the performance appreciably at the resonances; but may, if large, cause a few decibels loss as the

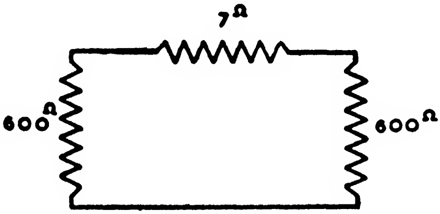


FIG. 47

cut-off frequency is approached in the case of a filter of several sections.

In the example worked out above, the main loss is a few ohms, namely 3·5Ω on each side of the line as it were joining the two end 600Ω resistances (see Fig. 47). The effect of this is to reduce the current in the ratio $\frac{1}{1200}$ to $\frac{1}{1207}$, neglecting the lattice losses, so the result is a current ratio $\frac{1207}{1200}$, which is of the right order as worked out in full to be 0·1 db.

The Use of Charts

Seeing that the attenuation is given in the form cosh P or tanh $\frac{P}{2}$, what is wanted is a means of finding P as a complex number when the right-hand side, which is the value of cosh P or tanh $\frac{P}{2}$, has been worked out from the circuit components.

In other words, given a and b in the equation cosh $(A + jB) = a + jb$, it is required to find A and B. Expanding cosh $(A + jB)$

gives $\cosh A \cos B + j \sinh A \sin B = a + jb$. Reals and unreals may now be equated.

$$\text{Cosh } A \cos B = a$$

$$\text{Sinh } A \sin B = b$$

The trouble is the two pairs of variables. If, however, B is assumed fixed for a time, i.e. given a particular value, a curve may be drawn

$$\text{Cosh } A = \frac{a}{\cos B}$$

$$\text{Sinh } A = \frac{b}{\sin B}$$

Then, as $\cosh^2 A - \sinh^2 A = 1$ always,

$$\frac{a^2}{\cos^2 B} - \frac{b^2}{\sin^2 B} = 1$$

which is like

$$\frac{x^2}{l^2} - \frac{y^2}{m^2} = 1$$

This is a hyperbola independent of A because A has been eliminated In other words, fix B and the values of a and b for various values of A lie on a hyperbola.

Similarly, if A is fixed the values of a and b as B is varied lie on an ellipse. This is the principle of Kennelly's charts, which contain instructions for use.

Circuit Components

It is difficult to make a good condenser, but a good mica condenser has a negligible loss, the main loss being then in the coil. Filter coils are usually wound on toroids of compressed iron dust and have a value of

$$\frac{\text{Reactance}}{\text{Resistance}}$$

of the order of 100 as against about 4 for stampings.

General Remarks on Losses

It is necessary to take particular formulas with particular arrangements of coils and condensers to work out the effect of losses in the components. The general formula for a lattice

$$\text{Tanh } \tfrac{1}{2}P = \sqrt{\frac{a}{b}}$$

is, naturally, no use by itself. One needs to know coils and condensers are in use in the a and b arms. Then, the formula for propagation constant must be worked out for a number of different frequencies throughout the pass band and the attenuation band.

In the pass band there is usually little loss introduced into the propagation constant except just near the cut-off, as a result of losses in the components. In other words, the pure phase change, which is the B in the unreal jB (which is what P reduces to in the pass band) without coil losses becomes $P = A + jB$ where A is a slight attenuation in népers. Good coils are used to minimize

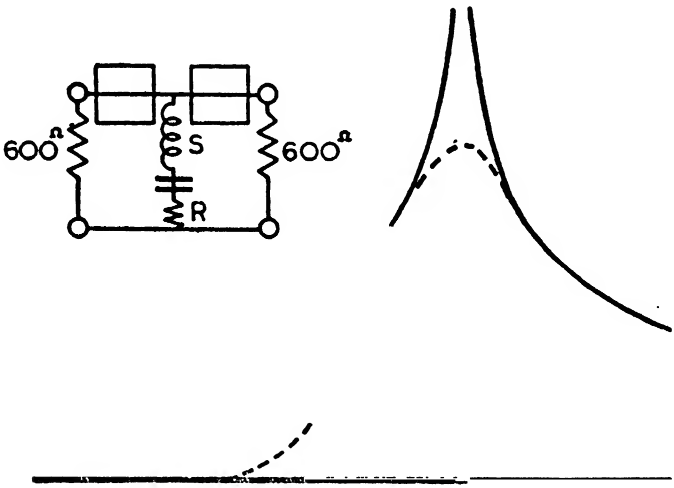


FIG. 48. THE EFFECT OF RESISTANCE R IS TO PREVENT THE SHUNT ARM S AT RESONANCE FROM FORMING A DEAD SHORT CIRCUIT ACROSS THE LINE AND SO CAUSING INFINITE ATTENUATION, WHICH IS THE CASE WITH NO LOSSES

the value of A. In the attenuation band the chief effect is at frequencies of infinite attenuation where the curve of attenuation no longer rises to infinity, but merely rises to a finite peak as shown in Fig. 48. (One néper is a voltage reduction of ε to 1, i.e. 2·71828, and two népers the square of this, and so on.)

In what has gone before, the calculated attenuations are those which would be observed if the filter section had been used in an infinite chain of similar filter sections, or at least in a chain of such sections that no reflections took place.

The calculated values of characteristic impedance, too, are also those values which would be measured at the beginning of an infinitely long chain of sections all similar, or, if not similar, all built

to have no reflections between sections (as the m-type sections are free from reflections between sections even when the sections are not all the same).

It may be objected, then, why use formulas for Z_0 (the characteristic impedance) and tanh P or cosh P when P and Z_0 do not refer to a practical case, i.e. to a case of just a few sections placed between apparatus which will not match the impedance of the filter when it is infinitely long? There are three good answers to this objection. The first is that reflections make the problem so complicated as to appear almost impossible of solution with the reflections in. Secondly, the calculated attenuations, i.e. values of P found on the assumption of an infinite chain, are very nearly equal to those found in practice in the pass band, and usually only a few decibels different in the attenuation band; so there is no need to jump out of the frying pan into the fire and use the more exact formulas for the end reflections in studying each type of filter circuit for the purpose of learning how each one behaves.

Lastly, the best way to develop an exact formula for the current at the far end of a filter in a circuit is to take the infinite filter first and calculate P and Z_0, then use that information together with the transmission line formulas that Heaviside developed for telephone and telegraph problems. These are to be found in Hill's *Transmission*, and in most books of a similar nature.

The propagation constant P and the characteristic impedance Z_0 of the filter are the "bricks" to use in building the house. The problem, then, is somewhat as follows: A certain circuit has to have a filter put in it. It must attenuate certain frequencies by a certain amount. If a ladder filter is decided on, one looks at the attenuation curve for one section of the ladder. It may give, say, 20 db loss one octave above the cut-off, but one may want 50 db at that frequency.

In that case about three sections are indicated as $3 \times 20 = 60$. The difficulty now is, how will the three sections behave when taken out of an infinite filter and put between resistance ends?

In the top portion of Fig. 41, page 65, an infinitely long filter is shown. In the lower portion of the same illustration three sections are taken out to be put in a circuit. The left-hand side may be an oscillator, the right a loaded cable.

The propagation constant in Fig. 48 and those given by all the previous formulas refer to the case of an infinite chain of sections, as in Fig. 41, by the nature and definition of a propagation constant, which is "the natural logarithm of the current and the voltage ratio

the 600^{Ω} ends without the filter in, i.e. $\frac{1}{1200^{\Omega}}$, gives the loss as a current ratio, which may be turned into decibels.

This is an easy formula to handle; for cosh P is often calculated in the normal course of events, and its value is available to go into

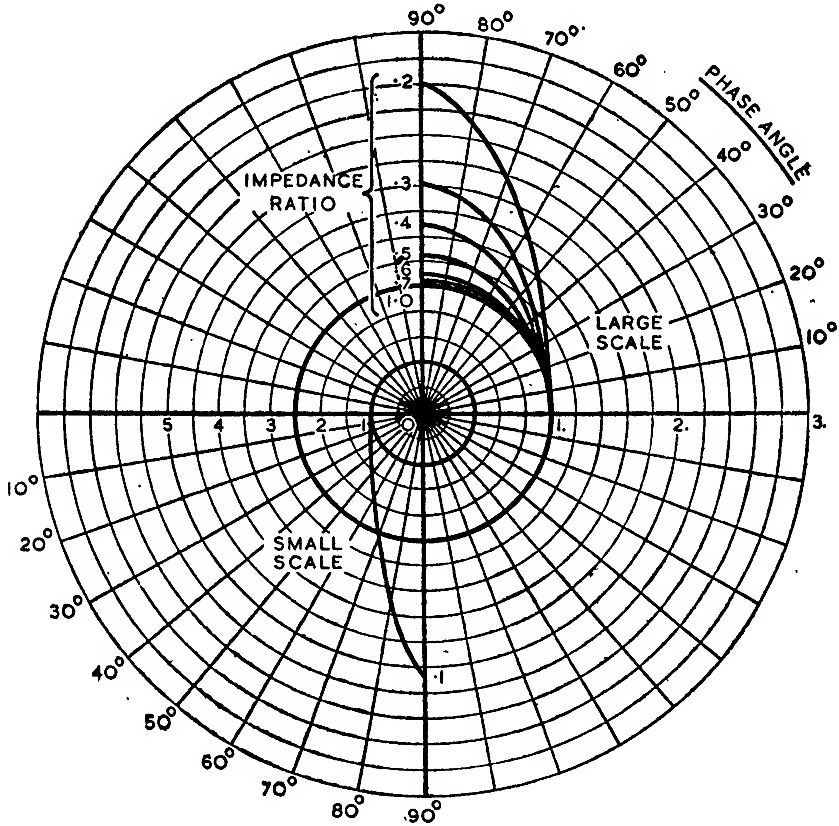


FIG. 50. INSERTION LOSS DIAGRAM FOR THE PASS BAND

the formula straightaway. Also sinh P is easily found from cosh P,

As an example, take a single section of the gate low pass filter. good impedance type. Its impedance and phase change in the pass band, neglecting coil losses, are as shown on page 84.

The loss may be found from a special chart.

The Use of the Chart (Fig. 50)

The chart is in two parts as the portion for an impedance mismatch of 1 : 10, i.e. 0·1 ratio, had to be drawn to a reduced scale. Cos $B + j \sin B$ is a unit vector whose end lies on the circle of unit radius. If, now, one increases the $j \sin B$ in the ratio

$$1 : \frac{\phi + \frac{1}{\phi}}{2}$$

the result is another vector whose end lies on an ellipse. (It is the same argument as that used in making sun-dials.) So use the ellipse marked for the desired impedance mismatch, but enter the chart on the circle at that radius shown by the phase angle and move vertically up to the desired ellipse. When on the circle, one has unity radius showing zero loss whatever the phase change, but when moving up to an ellipse showing mismatch also, one is going farther away from the centre. One moves *vertically* upwards, but having got to the ellipse the new *radius* tells the loss expressed as a fraction.

The following figures are calculated for the low pass gate filter and can be used with the chart to find what insertion losses may be expected when such a filter is put between two resistance terminations.

$\frac{f}{f_0}$	$\frac{Z_0}{\text{Line Res. } R}$	Phase Shift per Section $B°$
0	1	0
0·6	0·97	$72\frac{3}{4}$
0·8	0·88	$106\frac{1}{2}$
0·9	0·714	130
1·0	0	180°

The method of using this information is first to turn to the table and find

$$\frac{\phi + \frac{1}{\phi}}{2} \text{ for values of } \frac{f}{f_0}. \text{ Thus—}$$

$\frac{f_0}{f}$	$\frac{\phi + \frac{1}{\phi}}{2}$
0·6	1 nearly
0·8	1 nearly
0·9	1·07

The only value for loss is 0·9 where we now evaluate

$$\sqrt{\cos^2 130° + 1{\cdot}07^2 \sin^2 130°}$$

or else use the chart of ellipses for losses in the pass band.

$$\begin{aligned} &(-0{\cdot}62)^2 + 1{\cdot}14\,(0{\cdot}766)^2 \\ &= 0{\cdot}384 + 1{\cdot}14 \times 0{\cdot}586 \\ &= 0{\cdot}384 + 0{\cdot}668 \\ &= 1{\cdot}052 \\ &\sqrt{1{\cdot}052} = 1{\cdot}03 \end{aligned}$$

which is about 0·25 db

The measured results show more loss than this. That probably means there were harmonics on the oscillator wave of a few per cent which the filter cuts off.

To use the chart, find the spot on the unit circle for the phase angle of the network. Now go vertically up to touch the proper ellipse for the impedance ratio. These ratios are numbered 0·1,

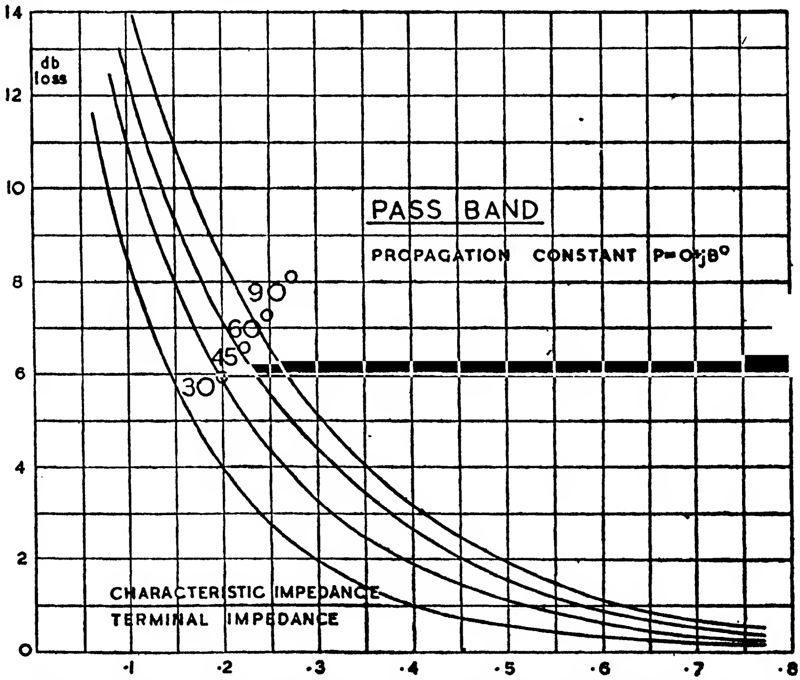


FIG. 51. THE TOTAL LOSS DUE TO THE INSERTION OF THE FILTER IN THE CIRCUIT (PASS BAND)

0·2, 0·3, 0·4, 0·5, 0·6, and 0·7. The point on the ellipse gives the end of the vector of current expressed as a ratio and also its phase angle.

EXAMPLE

A phase angle in the network of 76° and a mismatch of 2 : 1 gives 1·24 current ratio and about 80° phase difference between sent e.m.f. and received current. The chart applies only to the pass band. Figs. 51 and 52, however, take into account both the pass and attenuation bands; and from them it will be seen that the theorem given below is true.

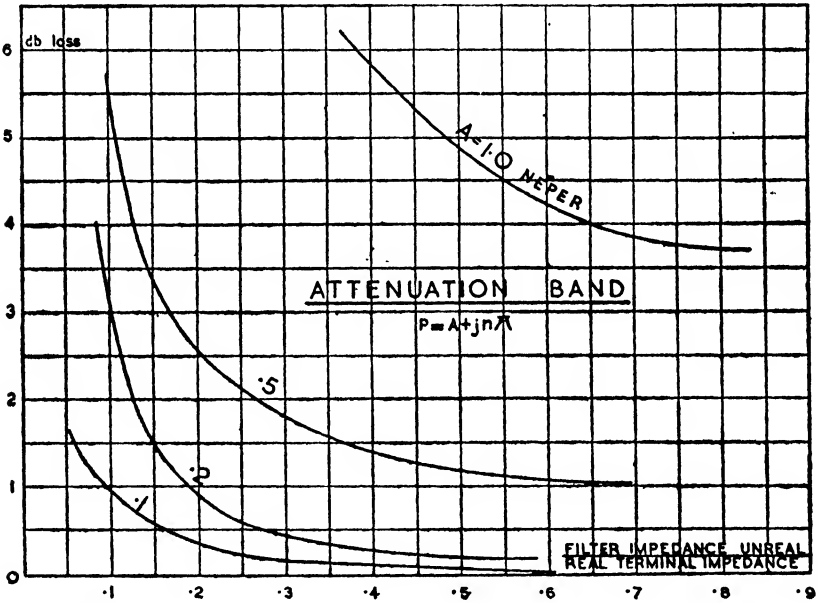


FIG. 52. THE TOTAL LOSS DUE TO THE INSERTION OF THE FILTER IN THE CIRCUIT (ATTENUATION BAND)

Theorem on the Effect of Terminal Reflections

When a filter is put into a circuit working between two resistances, reflections arise. These in general are adverse. They reduce the loss in the attenuation band and cause an unwanted loss in the pass band.

In the pass band where the filter itself has an internal phase change and where it matches the line more or less exactly the following theorem holds.

"Phase change in a filter in the pass band causes no loss if the impedance of the filter matches that of the terminations; and lack

of matching of these impedances also causes no loss, if there is no internal phase change. Phase change together with lack of matching causes loss."

The curves have been plotted from the telegraph equation.

It was soon found that the lattice filter—the easy ones at least—had the same attenuation curve and characteristic impedances as certain ladder filters, and this led to the idea that certain circuits looking very different may be very much or, rather, exactly alike in frequency performance.

CHAPTER V

THE EQUIVALENCE OF NETWORKS

CERTAIN four-terminal networks may be made which are exact equivalents of other four-terminal networks. Given a ladder section, for instance, it is possible to construct a lattice which shall have the same Z_0 and the same propagation constant. The two are thus equal as four-terminal networks, though one will usually be a simpler circuit than the other, and this is the one to be used in building up a particular circuit.

Ladder and Lattice Networks

It has been shown that the ladder network is equivalent to a lattice under the conditions shown in Fig. 53. Here the *a*'s and *b*'s

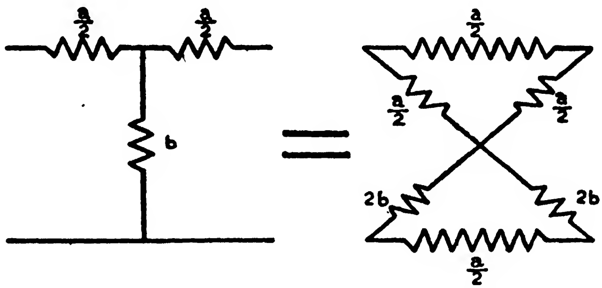


FIG. 53. LADDER AND LATTICE EQUIVALENTS

have the same meaning in both cases. If a is a one henry coil, $\frac{1}{2}a$ is a half henry coil. If a is a 1 μF condenser, $\frac{1}{2}a$, which must be half the impedance of a, is a 2 μF condenser. If b is a 0·1 μF condenser then $2b$ is a 0·05 μF condenser, and so on, as stated in Theorem IV of Chapter IX.

It will be seen that it is always possible, given a ladder network with its coils and condensers, to make a lattice equal in all respects, because the one lattice arm $a/2$ is a copy of part of the ladder and the other lattice arm, being $a/2 + 2b$, is two impedances in series. The lattice may, however, be wasteful, i.e. have more components than a ladder would. The ladder-lattice equivalent is found from Z_0 for the ladder being $\sqrt{Z_1Z_2 + \frac{1}{4} Z_1^2}$, but $\sqrt{ab}$ for the lattice. These need to be considered together with the attenuations to prove the above rules for making lattices from ladders.

Starting with a particular lattice, with its special coils and condensers, it is not always possible to make a ladder equivalent because the shunt arm of the ladder is to be found by subtracting the two

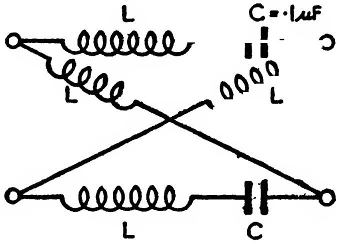


FIG. 54

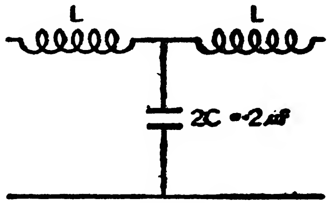


FIG. 55

lattice arms. This may not be easy if they are complicated circuits with coils and condensers. In certain cases the result is, however, very simple indeed as may be seen in the following example. Suppose the lattice is as shown in Fig. 54. Subtract the two arms and divide by 2 to find b. Subtraction gives 0·1 μF and division of impedance by 2 gives 0·2 μF. The ladder is then as shown in Fig. 55, which is simpler than the lattice. The reader may complain that the wrong arms have been subtracted here, but the lattice may be quite well re-drawn as shown in Fig. 56. A phase reversal is all that has happened, the crossing over of O and D.

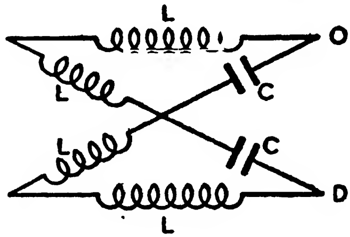


FIG. 56

The Lattice and the Gate Network

The gate is shown by the formulas for Z_0 and tanh P to be equivalent to the lattice which has "a and b in parallel" for one arm and "d and b in parallel" for the other, as shown in Fig. 57.

From this it will be seen that the gate is obviously superior to the lattice equivalent for the simpler circuits at least, because the b arm is used twice in the gate but four times in the lattice. Last of all, it may be shown that the gate network is equal to two ladder networks.

The Gate and the Ladder

By using the first equivalence for the lattice portion of the gate, and then adding the shunt b arms to the ends of the ladder, a ladder

equivalent to the gate may be made. This equivalent network is shown in Fig. 58.

Here a, d, and b are the arms of the gate. To form

$$\frac{d-a}{2}$$

it is necessary to subtract one arm of the gate from the other lattice one. This may leave a "negative inductance" in the equivalent

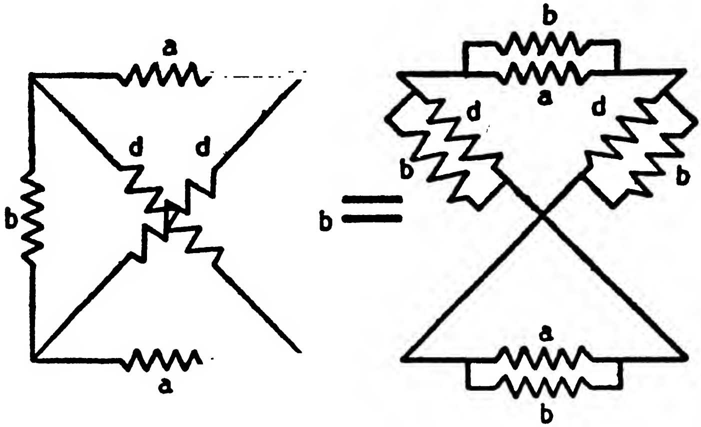


FIG. 57. THE GATE EQUIVALENT TO A LATTICE

ladder which could be realized by mutual inductance between the two a a coils of this ladder equivalent: but mutual inductance is not too easy to handle in practice, with the necessary inductance of each coil as well.

The sketch shows in a general way that the gate network is equal to two ladder sections, and is cheaper, needing fewer coils as a rule.

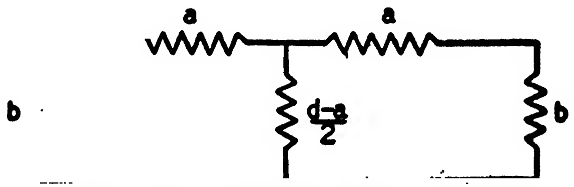


FIG. 58. THE LADDER EQUIVALENT TO THE GATE

It can be shown, too, that a ladder of several sections can be replaced by a single section having complicated arms. The outcome of this is

that filters can be designed in many ways. The simple ladder network has the merit of simplicity, but the two terminations trouble beginners. The gate network has no such complication and is superior in performance.

There is another network which may be equivalent to a lattice and which sometimes saves components. In any case, it is useful

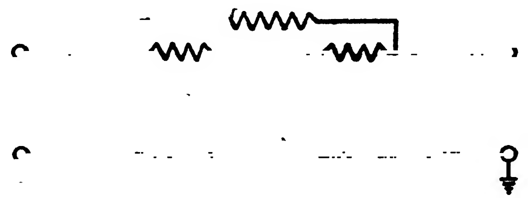


FIG. 59. BRIDGED "T" NETWORK

when one terminal of the input and one terminal of the output are earthed, as shown in Fig. 59. It is called the bridged "T" network. An example of this network is given later (see Fig. 63). Fig. 59 shows the way the extra resistance is added, forming a mesh.

Bridged " T " Networks

This network has a delta or mesh of impedance in it. If the mesh be replaced by a star or "T," then the whole thing becomes a simple "T" or ladder. In order to design the bridged "T" from the ladder,

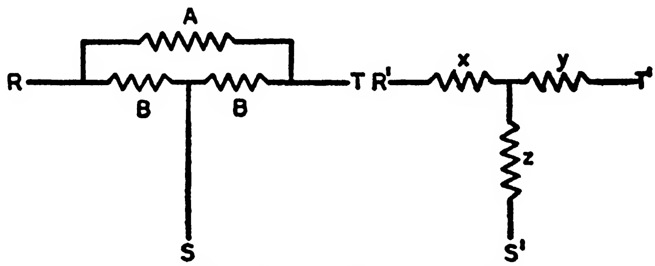


FIG. 60. STAR AND MESH NETWORKS

a theorem is needed which shows the relation between a star and a mesh. The requisite theorem is the "star mesh theorem." (See Figs. 60 and 61 for the solution to Fig. 59.)

The corresponding lattice to a ladder whose arms are a and b is a lattice whose arms are $\frac{1}{2}a$ and $(\frac{1}{2}a + 2b)$. The ladder which is equal

to a lattice having arms a and b is then one with full series impedance $2a$ and shunt impedance

$$\frac{b-a}{2}$$

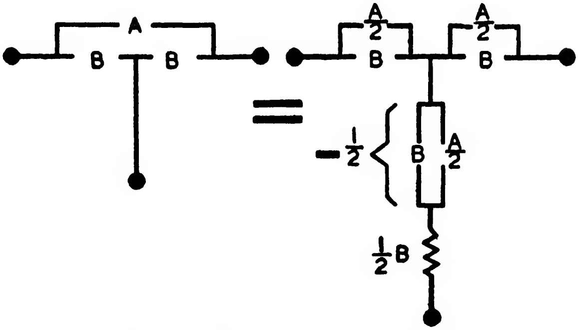


FIG. 61. THE STAR-MESH SOLUTION

The $2a$ full series impedance makes each arm of the equivalent "T" of value a. It may be impossible to construct

$$\frac{b-a}{2}$$

but it is possible to make a bridged "T" equivalent of the lattice network shown in Fig. 62, and this will be seen as follows.

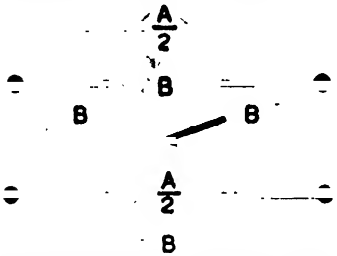


FIG. 62. THE LATTICE EQUIVALENT TO THE STAR AND MESH

The Star Mesh Theorem

The equivalence of the networks in Fig. 60 depends on certain relations between the arms of one, which we suppose may be anything, and the arms of the other. These relations are found by equating the resistance across the two terminals RS to that across R^1S^1. Then ST must be equated to S^1T^1 and RT to R^1T^1. In symbols this is

$$\frac{B(A+C)}{A+B+C} = x + z \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{A(B+C)}{A+B+C} = x + y \quad . \quad . \quad . \quad (2)$$

$$\frac{C(A+B)}{A+B+C} = y + z \quad . \quad . \quad . \quad . \quad (3)$$

Subtracting (1) from (2) gives

$$y - z = \frac{AC - BC}{A + B + C}$$

Using (3) gives

$$y = \frac{AC}{A + B + C}$$

Similarly

$$z = \frac{BC}{A + B + C}, \text{ and } x = \frac{AB}{A + B + C}$$

If now $x = y$ and $B = C$, which is the case for building a ladder from a bridged "T,"

$$x = y = \frac{AB}{A + 2B}$$

which is $\frac{1}{2}A$ in parallel with B. And

$$z = \frac{B^2}{A + 2B} = \frac{B^2 + \frac{1}{2}AB}{A + 2B}$$

$$- \frac{\frac{1}{2}AB}{A + 2B} = \frac{1}{2}B - \frac{1}{2}\left(\frac{\frac{A}{2}B}{\frac{A}{2} + B}\right)$$

The portion on the right is minus half an impedance, which is ($\frac{1}{2}A$ and B in parallel), so that the circuits shown in Fig. 61 are

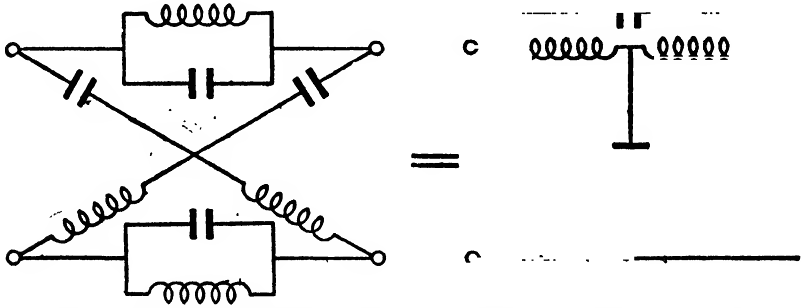


FIG. 63. THE LATTICE EQUIVALENT TO A USEFUL BRIDGED "T" PHASE CORRECTOR NETWORK

equivalent. The value of this lies in the fact that the lattice equivalent of the ladder in Fig. 60 has simple positive arms, as shown in

Fig. 62, because the series arm of the "T" added to twice the shunt arm makes two negative and positive impedances add up to zero. The lattice in Fig. 62 is not, however, a constant resistance network, because one arm is not quite the inverse of the other.

If, however, $\frac{1}{4}A$ is added to the shunt arm of the ladder and bridged "T" in Fig. 61, these become equal to the lattice shown in

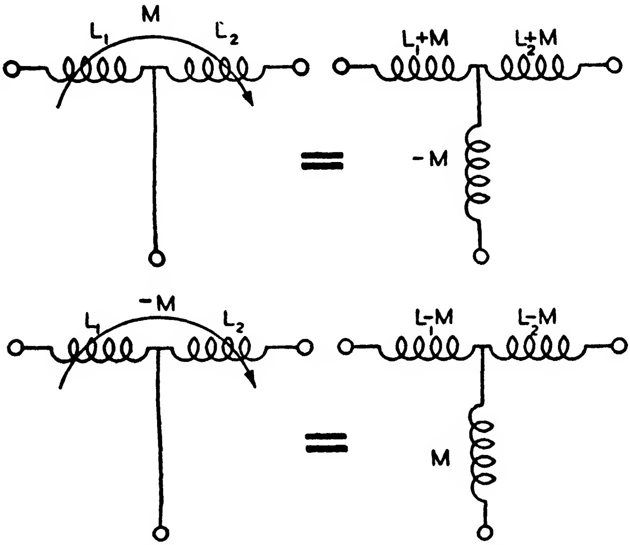


Fig. 64. A Useful Equivalent Circuit for the Case of Mutual Inductance Between Two Coils

Fig. 63, which is a constant resistance network if A is a coil and B a condenser.

This makes a very simple bridged "T" equivalent to a useful but fairly complex lattice. There are certain simple bridged "T" filters which have a very sharp cut-off.

The Use of Mutual Inductance in Filter Design

Filters with mutual inductance are most easily treated by looking at them as equivalent circuits to certain others which contain no mutual inductance.

The two pairs of equivalences in Fig. 64 are easily proved. When two coils of inductance L_1 and L_2 are connected in series aiding each other, M being the mutual inductance between them, the inductance in series is $L_1 + L_2 + 2M$. In opposition it is $L_1 + L_2 - 2M$.

The essential thing is that in Fig. 64 the inductance measured for each pair of terminals in the one drawing must equal that measured across the two corresponding terminals in the other or equivalent sketch.

Mutual Inductance in Lattices

When four inductances in two pairs of equal windings for a lattice are wound on the same core, there are six mutual inductances. These are all allowed for, if the inductance round one loop A_1A_1

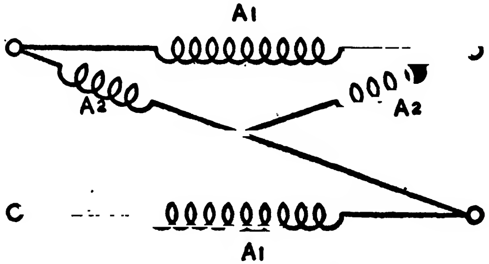


FIG. 65. THE MUTUAL INDUCTANCE OF LATTICES

Loop inductance of A_1A_1 windings = L_1 henrys
Loop inductance of A_2A_2 windings = L_2 henrys

is called "L_1 loop," and if the inductance round the other loop A_2A_2 is called "L_2 loop," and if M is defined as the mutual inductance between the loop of inductance L_1 and the loop of inductance L_2. The sketch Fig. 65 refers to this case.

The equivalents mentioned above are very useful in designing networks. The last but one, i.e. Fig. 64, enables ladder networks with mutual inductance to be built up to give the effect of negative inductance. The one in Fig. 63 enables a useful bridged "T" network to be built up for phase shifting.

Before going farther, it might be as well to mention the tuned transformer so much used in radio engineering in the design of "superhets." It can be shown that the circuit is equivalent to a well-known lattice band pass filter. This is done as follows.

The Tuned Transformer Band Pass Filter

In superheterodyne receivers, the intermediate frequency filter is usually a transformer with small condensers across primary and secondary as shown in Fig. 66 (a).

The advantages are that it is quite a simple arrangement and that it keeps the H.T. from the plate of the previous valve off the grid of the next one, and with so little apparatus makes a band pass filter. In its function as a band pass filter the circuit may show one peak or two peaks on the voltage/frequency graph of its performance, depending on the coupling of the coils, the k that is, where $k = \dfrac{M}{\sqrt{L_1L_2}}$. When the coils are equal on the primary and secondary sides then $k = \dfrac{M}{L}$ because L_1L_2 becomes L^2, and $\dfrac{M}{\sqrt{L^2}}$ becomes $\dfrac{M}{L}$, so k has a simple form.

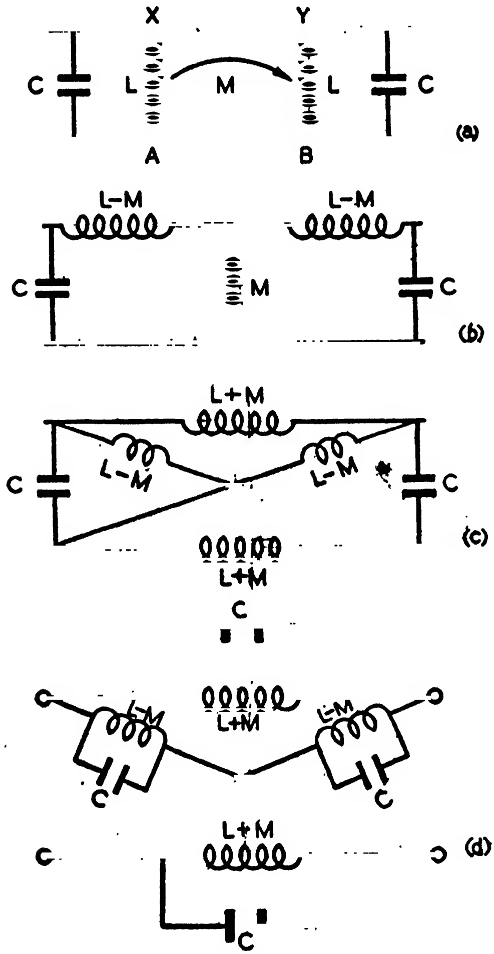


FIG. 66. LATTICE EQUIVALENT OF A TUNED TRANSFORMER BAND PASS FILTER

This is a difficult circuit to analyse and it is not easy to show that there is a filtering action. It has been worked out by long methods using the j notation, but the present material is written to show that it is a filter in a way that uses only a few simple circuit transformations.

This circuit and the grid leak detector are the two difficult spots in radio theory. All the rest, though it may be long, consists of simple theory (such as $V = jL\omega I$ for a coil and $I = jVC\omega$ for a condenser) used over and over again with Ohm's Law. The mutual inductance makes this circuit hard to understand.

In filter design a mutual inductance is often replaced by the circuit illustrated in Fig. 66 (*b*). The circuit change shown consists of earthing or commoning the points A and B. The condensers are left in because they are in the original tuned transformer circuit. To show that Fig. 66 (*b*) with its three separate coils $(L - M)$, M and $(L - M)$ forming a "T" is a true equivalent of Fig. 66 (*a*) (where the coils $L_1 L_2$ are coupled magnetically with mutual inductance M) notice that if the condensers are removed, measurements on terminals X and A give L henrys in Fig. 66 (*a*) and $L - M + M$, which is also L henrys, in Fig. 66 (*b*). Similarly for terminals Y and B the circuits are truly equivalent. Suppose also the tuning condensers are equal.

The next step is to form the equivalent lattice to the "T." Given a "T," one makes an equivalent lattice by taking one arm equal to half the series arm of the "T," which is in this case $(L - M)$, and taking for the other arm the $(L - M)$ with double the shunted M or $(L - M + 2\,M)$ giving $L + M$. The lattice is now as shown in Fig. 66 (*c*).

It is a general rule in four-terminal networks to consider the impedances at one end with the far end closed and also open. Two tests are enough.

It is also known that Fig. 66 (*c*) is equivalent to Fig. 66 (*d*). A gate may always be replaced by a lattice in that way, by shunting the end impedance across all four lattice arms.

The result is a lattice which is recognized as a band pass filter. The arm $(L + M)$ with C across resonates at a lower frequency than the arm $(L - M)$ with C across, simply because $(L + M)$ is bigger than $(L - M)$ henrys.

Further, at low frequencies the condensers do not count; and there will be little voltage built up across terminals Y and B with a given current through terminals A and X, which current usually comes from a valve of the pentode type, and is constant whatever the impedance in the plate circuit.

At high frequencies the coils will admit little current, and the condensers in Fig. 66 (*d*) give a nearly perfect Wheatstone balance, because the four C's in this diagram, which is an exact equivalent of Fig. 66 (*a*), are all equal.

The condition for a pass band in Fig. 66 (*d*) is that one arm, which in this case is one tuned circuit, must behave like a coil (we must therefore be below the resonant frequency of that arm) since a parallel coil and condenser admits a "coil current" below resonance. The other circuit must behave like a condenser (which it will do if

we are working at a frequency above the resonant frequency of that arm). The condition is, then, that we must work at a frequency between the resonant frequencies of the arms $(L + M)C$ and $(L - M)C$ or between

$$\omega_1 = \frac{1}{\sqrt{(L+M)C}} \text{ and } \omega_2 = \frac{1}{\sqrt{(L-M)C}}$$

Since $M = Lk$ we have

$$\omega_1 = \frac{1}{\sqrt{LC(1+k)}} \text{ and } \omega_2 = \frac{1}{\sqrt{LC(1-k)}}$$

Call ω_0 the frequency of resonance of L and C, i.e. when $M = 0$ or when the coupling in the original transformer is nil. Then

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ and } \frac{\omega_1}{\omega_0} = \frac{1}{\sqrt{1+k}} \text{ and } \frac{\omega_2}{\omega_0} = \frac{1}{\sqrt{1-k}}$$

Now use wavelengths as reciprocals of frequencies in the ratios above, and

$$\frac{\lambda_0}{\lambda_1} = \sqrt{1+k} \text{ and } \frac{\lambda_0}{\lambda_2} = \sqrt{1-k}$$

Then

$$\frac{\lambda_0^2}{\lambda_1^2} = 1 + k; \; \frac{\lambda_0^2}{\lambda_2^2} = 1 - k$$

So adding,

$$\frac{\lambda_0^2}{\lambda_1^2} + \frac{\lambda_0^2}{\lambda_2^2} = 2$$

If λ_1 and λ_2 are nearly equal to each other and so close to λ_0 it follows that $\lambda_1\lambda_2 = \lambda_0^2$.

Since $\lambda_0^2\lambda_2^2 + \lambda_0^2\lambda_1^2 = 2\lambda_1^2, \lambda_2^2$ from multiplying the line before, then $\lambda_2^2 + \lambda_1^2 = 2\lambda_0^2$, or λ_2^2 is as much above λ_0^2 as λ_1^2 is below λ_0^2 since they add up to $2\lambda_0^2$. This is for a narrow band, which is the practical case.

It is established, then, that it is a band pass filter, and the two cut-off frequencies are established also, and a useful simple approximation is also found for the λs or wavelengths in the practical case of a narrow band.

When the primary is fed from a high impedance valve and a frequency run is taken, measuring the secondary open circuit voltage, a curious thing emerges. With biggish couplings, and a good space between λ_1 and λ_2, i.e. a band width not too narrow, as resonance of *either* the $(L + M)C$ or the $(L - M)C$ arm is approached the output

voltage rises. This is fairly obvious, because the resonance of one pair of arms of the lattice is equivalent to open circuiting two opposite arms of a bridge, and so unbalancing the bridge in a way leading to a high open circuit voltage across the output or "detector" terminals.

Last of all, in the case of a fairly wide band, the circuits of all four arms are well off resonance, and so well away from their maximum impedances when we are in the middle of the pass band. This indicates a dip in the response curve in the middle of the band (with a wide band) which is absent with a narrow band when a looser coupling is used. The advantages of two "peaks" are well known in the design of superheterodyne receivers.

CHAPTER VI

PHASE SHIFT NETWORKS

The Use of Phase Shifting Networks

FOUR-terminal networks containing reactances which are, in general, filters shift the phases of applied currents in the pass bands. In those cases where there is no attenuation at any frequency the network merely causes a shift of phase whatever the frequency, and is then called a phase shift network. Such networks are useful for correcting the phase shift caused by transmission lines and other apparatus.

They are made with a characteristic impedance a constant at all frequencies, and thus introduce no reflection losses when put in series with other resistance apparatus. If, then, only one end is terminated correctly, this is enough to secure no extra reflections and so the introduction of the network merely adds the unreal B or phase shift to the circuit.

The reason for this is as follows. Phase shift alone is not necessarily bad. Where the phase shift is proportional to frequency throughout the frequency range in use and where the graph of B against ω is a straight line through the origin, there is a delay of the signal, caused by the phase shift, but there is no distortion. Phase shift is bad when the angle of phase shift B is not proportional to ω.

It is $\frac{dB}{d\omega}$ which matters and which should be constant. In the low pass filter, Fig. 23, it will be seen that B in the pass band has a general tendency to a straight line through the origin. For this reason such filters pass signals with little distortion provided the frequencies in the signals are mostly below the cut-off of the filter.

Seeing, however, that there is a steeper piece of the $B - \omega$ curve near the cut-off, and seeing that lines with loading coils act as low pass filters, such lines can cause distortion due to the change of $\frac{dB}{d\omega}$ in the line with change of frequency near the cut-off frequency.

The result of this in transmitting pictures is that a lack of definition is caused. This may be improved by levelling up the line as regards $\frac{dB}{d\omega}$. In other words, at those frequencies for which the line has a low value of $\frac{dB}{d\omega}$ a network is designed to have a high $\frac{dB}{d\omega}$. The

network is then put in series with the line and it is found that improvement results by making $\frac{dB}{d\omega}$ more nearly uniform.

Design of Phase Networks

The simplest lattice phase shift network (see Fig. 67) is one in which the series arms are each a coil of inductance L and the cross

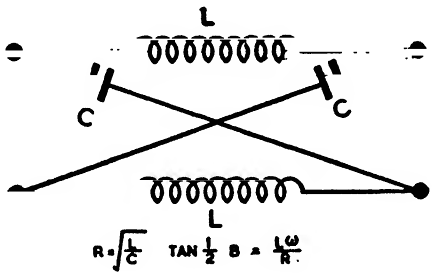


FIG. 67. SIMPLE LATTICE PHASE SHIFT NETWORK

arms each a condenser of value C, with resistances negligible. The lattice formulas are now used. Here $a = jL\omega$ and $b = \frac{1}{jC\omega}$ in the general lattice formulas. As $Z_0 = \sqrt{ab}$ the characteristic impedance is $\sqrt{\frac{L}{C}}$. The j's going out shows that it is a resistance, the ω's going out shows it is a resistance which is constant with frequency. This may be called R, and will match a line of R ohms. The phase change is given by $\tanh \frac{1}{2}P = \sqrt{\frac{a}{b}}$, and this is $\frac{a}{\sqrt{ab}}$. Since $\sqrt{ab}$ is a resistance R, this becomes $\tanh \frac{1}{2}P = \frac{a}{R} = \frac{jL\omega}{R}$ or $j \tan \frac{1}{2}B° = \frac{jL\omega}{R}$ or $\tan \frac{1}{2}B = \frac{L\omega}{R}$.

EXAMPLE

Suppose the network is to match a line of 600 ohms and is to give a shift of 90° at 800~. Then $\frac{B}{2} = 45°$ at 800~, and $\tan \frac{1}{2}B = 1 = \frac{L \times 5000}{600}$ from which $L = \frac{600}{5000} = 120$ mH. Also as $\sqrt{\frac{L}{C}} = 600$, the value of C is $\frac{120}{360{,}000}$ $= 0{\cdot}333\ \mu$F for each cross arm.

The phase shift produced by this network when terminated by 600^{Ω} is given by Table 16 below. The results are shown on the graph, Fig. 68.

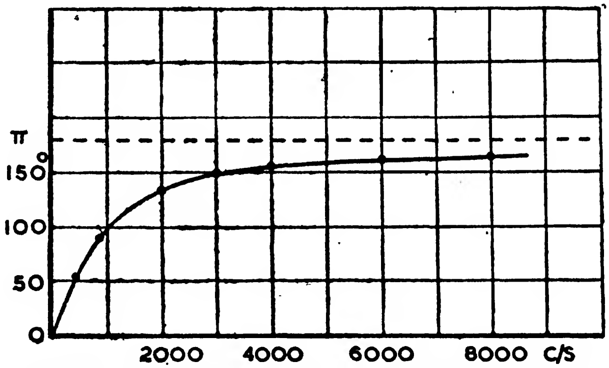


FIG. 68. GRAPH OF SIMPLE LATTICE PHASE NETWORK

TABLE 16

SIMPLE LATTICE PHASE SHIFT NETWORK

ω	$\frac{L\omega}{R} = \tan \frac{B}{2}$	$\frac{B}{2}$	$B°$
0	0	0	0
100	0·125	7° 7′	14° 14′
200	0·25	14° 3′	28° 6′
400	0·5	27°	54°
600	0·75	36° 54′	73° 48′
800	1·0	45°	90°
1000	1·25	51° 21′	102° 42′
1200	1·5	56° 21′	112° 42′
1500	1·875	61° 57′	123° 54′
2000	2·5	68° 12′	136° 24′
3000	3·75	75° 4′	150° 8′
4000	5	78° 42′	157° 24′
6000	7·5	82° 12′	164° 24′
8000	10	84° 18′	168° 36′
∝	∝	90°	180°

It will be seen that the steepness $\frac{dB}{d\omega}$ is first fairly constant, then falling off, just what is wanted to offset the changes in $\frac{dB}{d\omega}$ for the line.

Simple Lattice Phase Shift Network

In the case of the lattice phase corrector the lattice *b* arms must be inverse impedances with regard to the straight or *a* arms.

As described in Theorem VII (Chapter IX), the way to do this is to replace each coil of inductance L in the a arm by a condenser C such that $R = \sqrt{\frac{L}{C}}$ in the b arm, and each condenser C in the a arm by a coil L' in the b arm such that $R = \sqrt{\frac{L'}{C}}$. The resistance R is that for which the network is designed, i.e. the resistance to which it should be connected. In this, C is in farads as usual, L in henrys.

Where two components are in series in the a arm, the inverse components must be in parallel in the b arm. A parallel arrangement in the a arm gives a series arrangement in the b arm.

This is shown in Fig. 69. It so happens, however, that this network has a very simple bridged "T" form. The relation between the bridged "T" and the lattice will best be made clear by an example. It may be said that the inversion principle can be used, however complicated the a and b arms of the lattice may be.

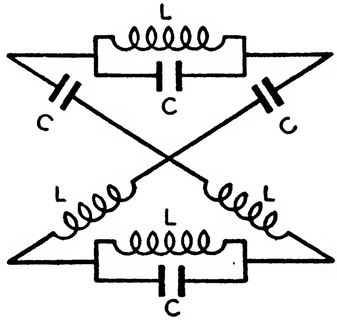


Fig. 69. A More Complicated Lattice Phase Corrector

Example

Suppose, then, a lattice network is designed to reverse the phase at 800 c/s and for a 600Ω line. Then $\frac{1}{f_0} = 2\pi\sqrt{LC}$ because at the resonance of L and C the current reverses and $\sqrt{\frac{L}{C}} = R$, since the one arm is an inverse impedance to the other.

Multiplying,
$$L = \frac{R}{2\pi f_0}$$
$$C = \frac{1}{2\pi R f_0}$$

In this example, $L = 120\text{mH}$, $C = 0{\cdot}33\mu\text{F}$.

Because of the simplicity of the lattice network this is the circuit to use in making calculations. The calculation is

$$j \tan \frac{B}{2} = \frac{\text{one arm}}{R}$$

The parallel arm is

$$\frac{\dfrac{0{\cdot}120}{0{\cdot}33 \times 10^{-6}}}{j0{\cdot}012\omega - \dfrac{j10^6}{0{\cdot}33\omega}}$$

Then as $R^2 = 360{,}000$

$$\tan\frac{B}{2} = \frac{-R^2}{R\left(\dfrac{10^6}{0{\cdot}33\omega} - 0{\cdot}12\omega\right)} = \frac{600}{\dfrac{10^6}{0{\cdot}33\omega} - 0{\cdot}12\omega}$$

When ω is small $\dfrac{10^6}{0{\cdot}33\omega}$ is very large, and B is small. However $\dfrac{B}{2}$ becomes 90° and B becomes 180° at resonance, i.e. at 800 c/s.

The phase now becomes negative and finishes at very high frequency a small negative angle, i.e. a slight lead. The current must

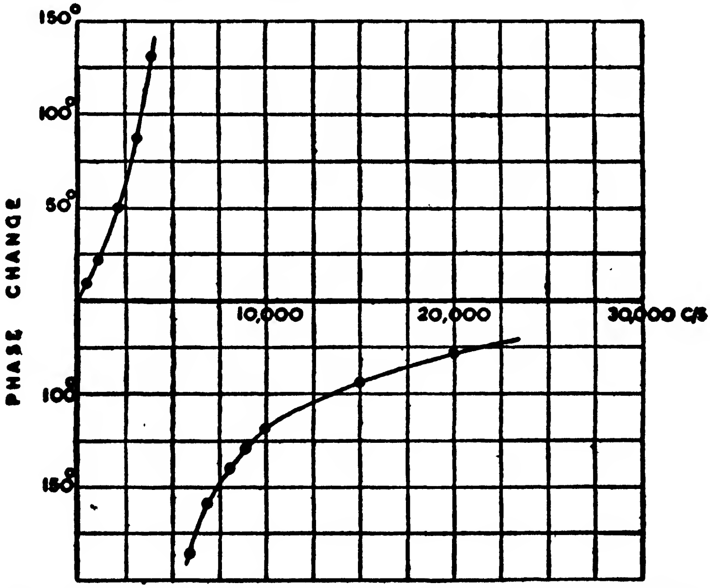


FIG. 70. PERFORMANCE OF BRIDGED "T" PHASE CORRECTOR CIRCUIT

lead at high frequency because it passes through the condenser in the parallel combination of coil and condenser. The arithmetic is as shown in Table 17.

TABLE 17

MORE COMPLICATED PHASE NETWORK

ω	$\frac{10^6}{0 \cdot 33\omega} - 0 \cdot 12\omega$	$\frac{600}{\text{same}}$ (tan $\frac{1}{2}B$)	$\frac{B}{2}$ (tables)	$B°$ (double $\frac{1}{2}B$)
500	6000 — 60	+ 0·101	5° 45′	11° 30′
1,000	3000 — 120	+ 0·208	11° 45′	23° 30′
2,000	1500 — 240	+ 0·476	25° 30′	51°
3,000	1000 — 360	+ 0·936	43° 15′	86° 30′
4,000	750 — 480	+ 2·22	65° 36′	131° 12′
5,000	600 — 600	∞	90°	180°
6,000	500 — 720	— 2·73	— 69° 54′	— 139° 48′
7,000	429 — 840	— 1·46	— 55° 36′	— 111° 12′
8,000	375 — 960	— 1·02	— 45° 36′	— 91° 12′
9,000	333 — 1080	— 0·803	— 38° 45′	— 77° 30′
10,000	330 — 1200	— 0·667	— 33° 30′	— 67°
15,000	200 — 1800	— 0·375	— 20° 30′	— 41°
20,000	150 — 2400	— 0·235	— 13° 20′	— 26° 40′
40,000	75 — 4800	— 0·126	— 7° 10′	— 14° 20′

(See Fig. 70 for graphical results of this lattice.)

In making the bridged "T" network to be equivalent to this lattice the 0·120 henry coil is the B of Fig. 60 and goes straight into the "T." The 0·33 μF condenser on the other hand is $\frac{A}{2}$ and the "T" circuit needs a shunt component $\frac{A}{4}$, which is a condenser 0·67μF, and a bridging impedance A, which is a condenser 0·167 μF. The actual bridged "T" is then as shown in Fig. 71.

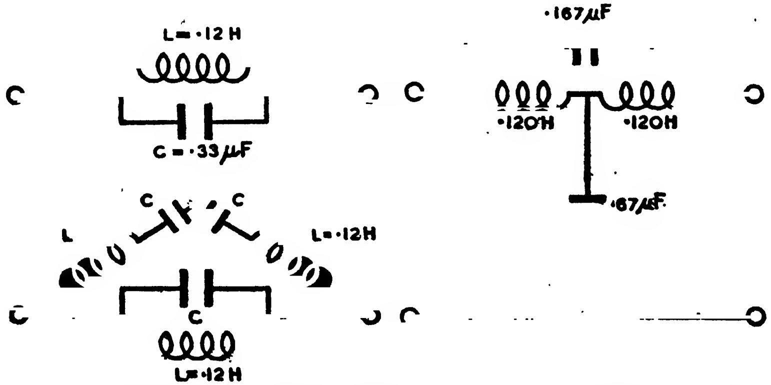


FIG. 71. EXAMPLE OF A USEFUL BRIDGED "T" PHASE CORRECTOR CIRCUIT AND THE EQUIVALENT LATTICE

There is an infinite variety of such networks possible, but they can be designed for any scale of frequency by altering L and C, making both larger or smaller. It is quite possible to make more complicated networks, but good results can be obtained with a number of lattices, as shown in Fig. 67 and in Fig. 69 all end to end. These should have the same value of resistance for their characteristic impedance, but may be different as regards frequency characteristics in their phase shift.

CHAPTER VII

ATTENUATION EQUALIZERS

It is desirable that all the frequencies of the voice on a telephone system should be received at the same strength relative to their strength at the sending end. Thus if 1 volt at 800 c/s at the sending end gives 10 millivolts at the receiving end, it is desirable that 1 volt at 2000 c/s should give 10 millivolts at the receiver also. In other words, the attenuation should not vary with frequency.

The attenuation of cables and lines always does vary with frequency and in carrier systems the band pass filters do not give quite

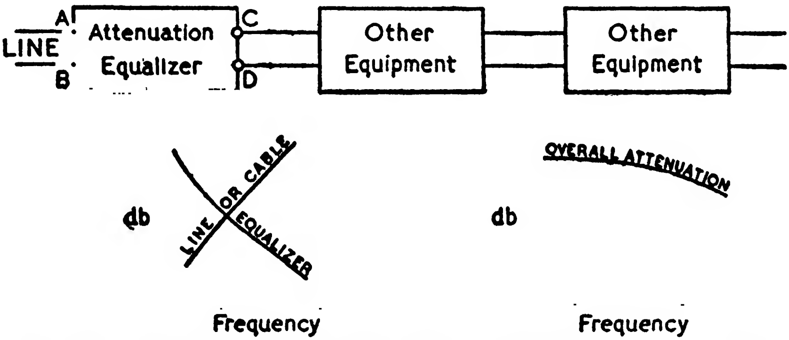


Fig. 72. The Position and Function of an Attenuation Equalizer

a constant loss through the pass band, not to speak of modulator and other circuits.

An attenuation equalizer is a circuit made up of coils, condensers, *and* resistances, so that its attenuation will vary with frequency in the opposite way to the circuit it is designed to correct.

Desirable Qualities for an Attenuation Equalizer

It should, in addition to providing the desired attenuation-frequency characteristic, possess an impedance looking into the terminals *AB* (see Fig. 72), which is a constant pure resistance when the terminals *CD* are connected to the resistance for which the equalizer was designed (600 ohms, say).

It is desirable that if the *AB* terminals are closed by this line resistance the impedance at the *CD* terminals should be a constant resistance, the same resistance *R* for the "line" in which it is

designed to be used. The "line," by the way, may be the 600^{Ω} resistance termination at the input of an amplifier.

In designing an equalizer for a channel of a carrier system it is usual to provide an equalizer which gives a sloping straight line something like—

25 kc	8 db
26 ,,	6 ,,
27 ,,	4 ,,
28 ,,	2 ,,

The equalizer should not have too much apparatus. For example, a lattice circuit in which the arm *a* is the inverse impedance of the

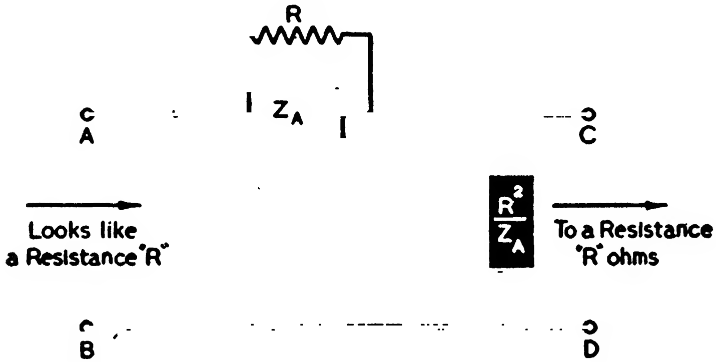


FIG. 73. CONSTANT RESISTANCE (ONE-WAY) ATTENUATION EQUALIZER

b arm is a good equalizer, but the bridged "T" is quite as good and has only two "impedance" arms to make up instead of four, though a lattice equalizer is balanced to earth if that is wanted, and the same coil will do for both the opposite arms in every case if double wound.

There are, then, three equalizers which are good, two because they are simple.

The Constant Resistance (One-way) "L" Equalizer (see Fig. 73)

This circuit, like the last, has a pair of impedances which are the inverse of one another. It differs from the last in three respects. The series arm is shunted by the line resistance *R*. This is a resistance spool made up to the figure *R*. When the shunt terminals *CD* are connected to the line, if this has the value *R* then the impedance looking into the *AB* terminals is a *constant resistance R* and the reflection loss between *CD* and the line merges with the attenuation proper into a total loss, which is given by a very simple formula

when the impedance of the series arm is known, and this loss may be read off on a simple chart (see page 113).

Proof of Constancy of Resistance for Constant Resistance "L" Equalizer

The proof that there is a constant resistance at AB when CD is closed with R is quite easy. Put $\frac{R^2}{Z_A}$ in parallel with R. This becomes

$$\frac{\frac{R^3}{Z_A}}{\frac{R^2}{Z_A}+R}$$

by the "product over sum" rule. Put Z_A in parallel with R and add the two.

$$\frac{\frac{R^3}{Z_A}}{\frac{R^2}{Z_A}+R}+\frac{RZ_A}{R+Z_A}$$

$$=\frac{R^2}{R+Z_A}+\frac{RZ_A}{R+Z_A}=R.$$

There is thus a constant resistance that way, i.e. looking into AB when CD is closed with the proper resistance R for which the circuit has been designed.

The Voltage Reduction Ratio of this Circuit

It is like a potentiometer where the total resistance across AB is R and the part tapped off is $\frac{R^2}{Z_A}$ with the line R in parallel. That is,

$$\frac{\frac{R^3}{Z_A}}{\frac{R^2}{Z_A}+R}=\frac{R^2}{R+Z_A}$$

The ratio $R : \frac{R^2}{R+Z_A}$ is the voltage ratio $AB : CD$. It is

$$\frac{R+Z_A}{R}=1+\frac{Z_A}{R}$$

This is an easy formula. Let $\frac{Z_A}{R}$ be written as a vector $r\underline{/\theta}$. Then in Fig. 74 the $1 + \frac{Z_A}{R}$ is a line OA. Therefore, a length OA of 1·5 means a voltage (and current) ratio of 1·5 : 1. This makes the chart a series of arcs of circles and the axes $O'X$ and $O'Y$ may be graduated in resistance and reactance of $\frac{Z_A}{R}$. Hence the distance $OO' = 1$ is the value of the line R as it makes $Z_A = R$ when $\frac{Z_A}{R} = 1$.

There is an interesting thing about this equalizer. If the resistance on the right-hand side happens to be a filter whose resistance is not constant and so is not correctly joined to the CD terminals of the equalizer, and if the line on the left is the proper resistance R, then the transmission from *right* to *left* will have the additional total loss indicated by the chart.

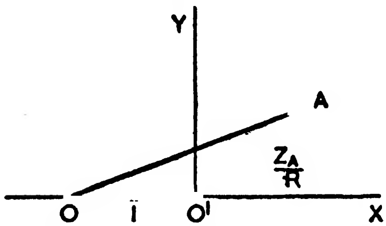


Fig. 74. The Voltage Ratio of the Circuit in Fig. 73

The L equalizer $= 1 + \frac{Z_A}{R}$

In view of the reciprocal theorem, however, the *left* to *right* transmission is the same as the right to left in milliamps. per generated volt.

With this type of equalizer, then, if there is only one impedance of the two which is a constant resistance where it is to be inserted it should be turned so that the series arm faces the constant or more constant resistance.

Design of Arm

It is best to take a network, say a series resonant circuit with a shunted resistance and shunted condenser, and obtain a few values of loss from the chart with different impedances. It is then possible to specify these impedances at, say, three or four frequencies, which finds the coil and condenser values in the Z_A circuit (see Fig. 75).

As mentioned above, when neither impedance looking away from the equalizer is a constant

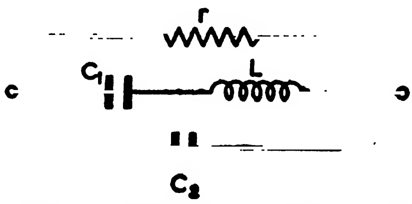


Fig. 75. A Typical Circuit for the Arm Z_A of an "L" Constant Resistance (One-way) Equalizer

resistance the chart figures will not be a correct loss, for there will be an additional reflection not taken into account by the chart.

Simple Example of the "L" Constant Resistance (One-way) Equalizer

Suppose the arm Z_A is 0·1 henry in series with 300 ohms. The resistance and reactance components of this arm are

as shown in the following table.

TABLE 18

ATTENUATION EQUALIZER

f	Resistance	Reactance	Resistance as a fraction of 600 ohms	Reactance as a fraction of 600 ohms	Voltage Ratio from Chart
200	300	125	0·5	0·204	1·54
400	300	250	0·5	0·417	1·56
800	300	500	0·5	0·833	1·72
1600	300	1000	0·5	1·667	2·24
3200	300	2000	0·5	3·333	4·16

The Construction of the Inverse Arm

A coil and resistance in series inverted becomes a condenser and resistance in parallel. The resistance makes 300 multiplied by it become $R^2 = 360{,}000$ ohms if $R = 600$ ohms, so the shunt resistance is 1200 ohms. The condenser is $\frac{L}{C} = R^2$, so $\frac{0{\cdot}1}{C} = 360{,}000$, and $C = \frac{1{\cdot}0}{3{,}600{,}000} = 0{\cdot}278\ \mu F$.

This makes the whole equalizer as in Fig. 76.

Constant Resistance (Both Ways) Attenuation Equalizers

The following bridged "T" circuit has an advantage over the "L" type in that it has constant resistance both ways. At the same time it is not as easy to calculate the loss with this equalizer, so it is doubtful whether the extra work in calculation is worth the slight risk of unknown reflection loss which may come in with the "L" equalizer.

A Chart for the "L" Equalizer

The formula

$$\text{Voltage ratio} = 1 + \frac{Z_A}{R}$$

is a simple one: it involves no hyperbolic functions, and has an easy pictorial representation.

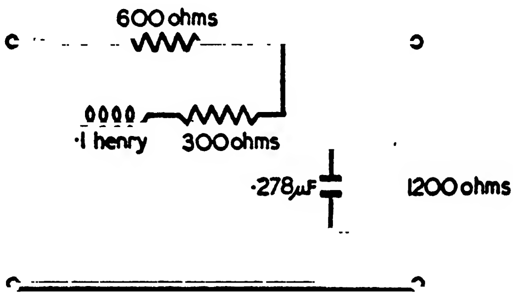


FIG. 76. A SIMPLE CONSTANT RESISTANCE (ONE-WAY) ATTENUATION EQUALIZER

The R for the line being a pure number like 600 ohms makes $\frac{Z_A}{R}$ easy to work out. As it is a vector, it is simple to affix 1 to it by addition. The length of the combined vector is the voltage ratio. Since the line is supposed to be R and the equalizer "looks like" R when it is closed by R, equal currents are obtained by equal voltages. Therefore the voltage and current ratio are equal.

The Construction of the Chart

It is convenient to make the chart suitable for $\frac{Z_A}{R}$ expressed in the $a + jb$ form of "effective resistance" and "effective reactance" in Z_A.

Since R is usually 600 ohms the unit is put at 600 ohms. Then, if we take a length equal to 600 ohms on the left of the origin this will be 1. Draw circles with this centre and radius greater than 1. These are the addition of 1 and $\frac{Z_A}{R}$. Their length indicates voltage and current ratio (see Fig. 77).

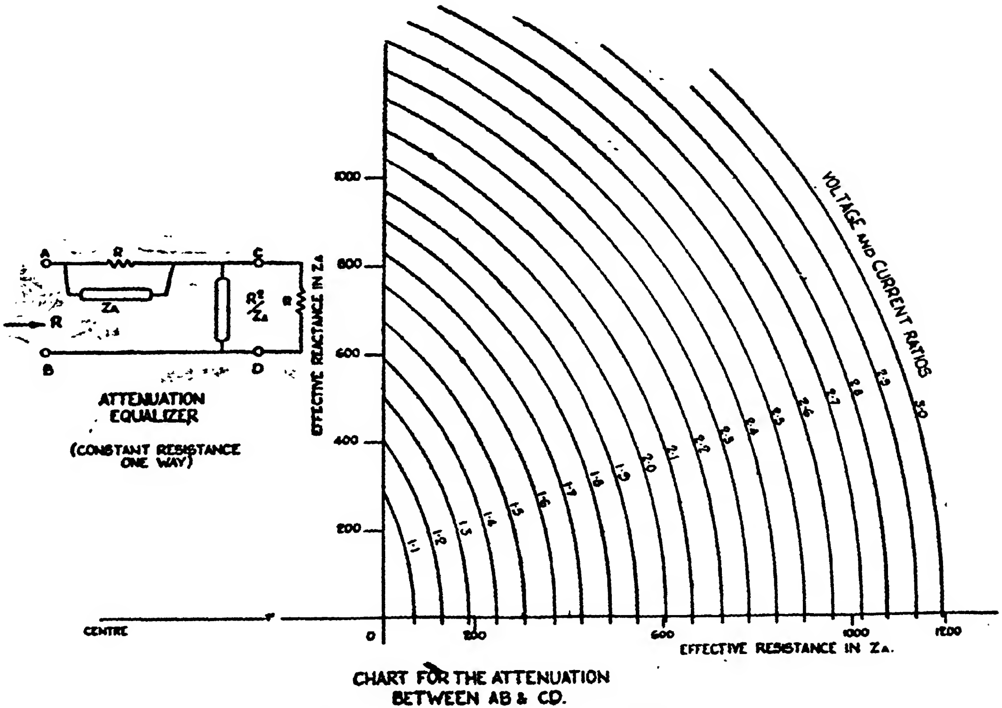


Fig. 77. Voltage and Current Ratio

CHAPTER VIII

MEASUREMENTS

Bridges and Measurements

THERE are two classes of measurements required in dealing with four-terminal networks. One is measurements made on components, i.e. coils and condensers; the other is measurements made on complete networks.

The first class of measurements involves two-terminal apparatus, and is carried out by bridges. There is a variety of bridges, but the half-dozen or so which are of greatest practical importance are treated in the following pages.

The most important A.C. bridge from the purely theoretical point of view was the mutual inductance bridge, for the following reason. If no apparatus of known value were available at all, a square of wire could be put up next to another square, both being of known size as measured by a rule. The mutual inductance between the squares may be calculated by Maxwell's formulas. The result is a primary standard of mutual inductance from which other standard inductances may be calibrated. The Campbell mutual inductance bridge is made up in this way and is in wide use.

It has the disadvantage that when used as a Carey Foster bridge for measuring condensers it is hard to get the true answer for the losses in the condenser being measured. Also, a given mutual inductance, even with added balancing coils, will not measure very large inductances and very small ones as well.

A Condenser as Standard

On the other hand, nowadays a good variable condenser box makes a suitable standard with which to measure other condensers and coils, and the box may be checked against a clock as described later.

With ratio arms and a variable resistance box it will make four bridges at least, and these will measure a surprising range of coils and condensers as well as impedances such as transmission lines and other pieces of apparatus, even where the resistance component of the impedance is much larger than the reactance. This is impossible with many commercial bridges.

Bridge measurements are often difficult for a variety of reasons. One is that no single adjustment will obtain a balance as a rule.

Both the resistance *and* the reactance in the apparatus to be measured must be balanced by the bridge in order to secure the zero current in the detector.

Another thing is that the telephone, the commonest bridge detector, is at its best over a very limited range of frequency unless it be a Piezo crystal telephone, which is roughly constant from 200–4000 c/s.

The Detector

The Duddel vibration galvanometer is excellent from the point of view of being a tuned detector and therefore not responding to harmonics in the oscillator or in the currents flowing in the apparatus.

Possibly the best arrangement for a detector is a visual one, either a voltmeter or a "visual tuning indicator" of the "magic eye" or similar type. The use of a low pass filter leading to the detector gets rid of any harmonics on the oscillator wave and also any harmonics due to the presence of iron in the apparatus under test when this is a coil with an iron core, or when it contains such a coil.

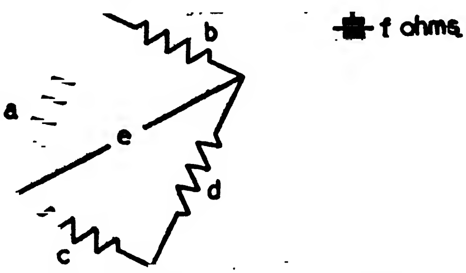


FIG. 78. THE WHEATSTONE BRIDGE

Coils with dust cores do not usually cause this trouble.

There is still another difficulty, however, and that is that the sensitivity of a bridge, as defined by the smallest change in the impedance under test which can be observed in the detector, depends on the impedances of the bridge arms and those of the oscillator and detector.

Impedance Relation in Bridges

Heaviside shows in his *Electrical Papers*, Vol. I, that the ratios a and b for greatest sensitivity in the bridge shown in Fig. 78 are given by

$$a = \sqrt{ef} \quad b = \sqrt{de\left(\frac{d+f}{d+e}\right)}$$

When oscillator, detector and all four arms are equal the *very* best result is obtained, but in general this is not the case as the impedances to be measured in light current work commonly vary

from a few ohms to about a megohm, and the detector cannot be altered easily.

For example, a 1 mH coil has a reactance of 5 ohms at 800 c/s; while a 100 henry choke at 1600 c/s is about a million ohms.

For this reason it is as well to use the best ratio which can be formed, according to the previous formulas.

With battery and galvanometer small resistances, $b = \sqrt{de}$ nearly.

Proof to the Negative Series Bridge

First write $\frac{1}{Cp}$ for the impedance of a condenser. Multiply opposite pairs of arms, and equate. This is the result for the bridge shown in Fig. 79.

$$a\left(R + \frac{1}{C_s p}\right) = b\left(r + \frac{1}{Cp}\right)$$

So

$$aR + \frac{a}{C_s p} = br + \frac{b}{Cp}$$

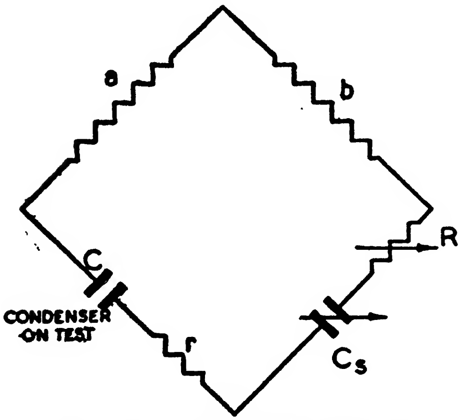


Fig. 79. Negative Series Bridge

The question to be asked is can one choose values for a, b, C and C_s and the resistances r and R, which will make the equation true? If so, the bridge will balance, that is to say, it is a practicable bridge. There are two cases where a bridge will balance. These are—

(1) Balance on any wave form whatever of whatever frequency.

(2) Balance with a sine wave of one frequency only.

Practically, if a bridge balances with Heaviside's unit function, i.e. square waves at a low frequency, it will balance on anything. Mathematically, if the operator p vanishes it will balance on anything, but if p^2 must be given a fixed negative value, that shows the bridge comes under Case 2 and will only balance with a sine wave, the frequency of which is given by $p^2 = -\omega^2$.

In the present case of a negative series bridge, if $aR = br$ and if also

$$\frac{a}{C_s} = \frac{b}{C}$$

then the equation balances whatever value p may have. Hence for balance

$$r = \frac{a}{b}R \text{ and } C = \frac{b}{a}C_s.$$

Negative Shunt Bridge

This bridge is shown in Fig. 80. Here the standard arm has resistance R and capacity C_s in parallel, so it is

$$\frac{\text{Product}}{\text{Sum}}$$

which is

$$\frac{\dfrac{R}{C_s p}}{R + \dfrac{1}{C_s p}}$$

Then multiplying opposite arms to form the equation, and calling the condenser on test $r + \dfrac{1}{Cp}$, we have

$$\frac{a\dfrac{R}{C_s p}}{R + \dfrac{1}{C_s p}} = b\left(r + \frac{1}{Cp}\right)$$

$$\frac{aR}{1 + RC_s p} = \frac{b\,(rCp + 1)}{Cp}$$

This can be made to balance only by calling $p = j\omega$. So we have

$$\frac{aR}{1 + jRC_s\omega} = b\left\{r + \frac{1}{jC\omega}\right\}$$

Now rationalize the left-hand side and

$$\frac{aR\,(1 - jRC_s\omega)}{1 + R^2C_s^2\omega^2} = br + \frac{b}{jC\omega}$$

Equate reals, then equate imaginary quantities—

$$\frac{aR}{1 + R^2C_s^2\omega^2} = br$$

$$r = \frac{a}{b\,(1 + R^2C_s^2\omega^2)}$$

giving the resistance of the condenser C. The imaginaries give

$$\frac{aR^2C_s\omega}{1 + R^2C_s^2\omega^2} = \frac{b}{C\omega}$$

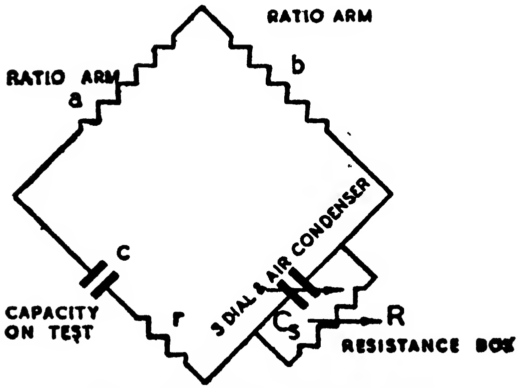


FIG. 80. NEGATIVE SHUNT BRIDGE

So
$$C = \frac{b}{a} R^2C_s\omega^2 \{1 + R^2C_s^2\omega^2\}$$

$$= \frac{b}{a} C_s + \frac{b}{a\,R^2C_s^2\omega^2}$$

which is the required capacity.

The next is a difficult looking bridge with an easy solution and easy formulas.

Positive Shunt Bridge

This is Maxwell's bridge if the condenser travels along the resistance R, shunting a part of it only. It is hard to arrange the tapping on a four-dial decade resistance box, and a slider is not too accurate, though it is used in bridges on the market. If the condenser C_s is variable, then the condenser can be kept in parallel with all the resistance in R and both varied (see Fig. 81).

Write L as Lp and cross-multiply

$$\{r + Lp\}\left\{\frac{R}{1 + RC_s p}\right\} = ab$$

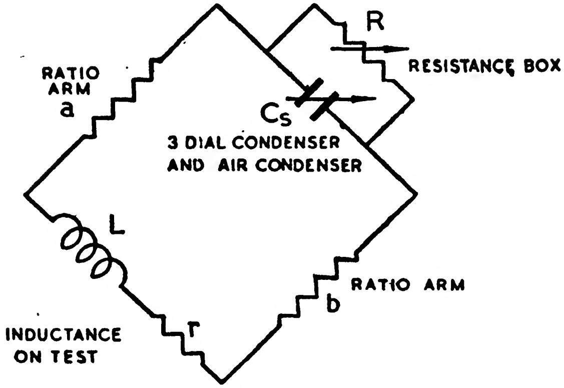


FIG. 81. POSITIVE SHUNT BRIDGE

This will be true for any value of p if

$$L = abC_s$$

and

$$r = \frac{ab}{R}$$

The bridge therefore balances with any frequency and wave form. If C_s is in microfarads, $a = b = 1000^{\Omega}$, and L is in henrys.

This last bridge is not easy to use with a good coil, such as a telephone loading coil, for the coil resistance r being low makes R need to be higher than one usually finds in a box.

Positive Series Bridge

The positive series bridge is a good one (see Fig. 82). Here

$$ab = \left\{R + \frac{1}{C_s p}\right\}\{r + Lp\}$$

The bridge requires a sine wave of any frequency. Put $p = j\omega$.

$$ab = \left\{R - \frac{j}{C_s\omega}\right\}\{r + jL\omega\}$$

$$ab = rR - \frac{jr}{C_s\omega} + jRL\omega + \frac{L}{C_s}$$

Equate reals and unreals

$$ab = rR + \frac{L}{C_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$RL\omega = \frac{r}{C_s\omega} \quad \therefore \ L = \frac{r}{RC_s\omega^2} \quad . \quad . \quad . \quad . \quad (2)$$

Put this in (1) and

$$ab = rR + \frac{r}{RC_s^2\omega^2} = r\frac{(1 + R^2C_s^2\omega^2)}{RC_s^2\omega^2}$$

$$\therefore \ r = \frac{abC_s^2\omega^2R}{1 + C_s^2\omega^2R^2}$$

for the coil resistance.

Also from (2) we have

$$L = \frac{abC_s}{1 + C_s^2\omega^2R^2}$$

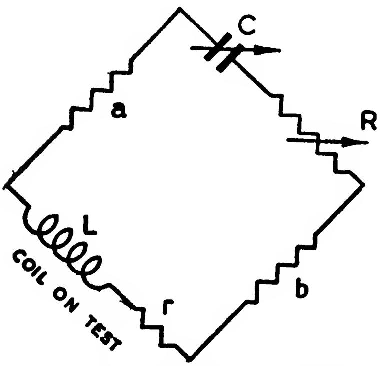


Fig. 82. Positive Series Bridge

for the coil inductance. Happily, $1 + C_s^2\omega^2R^2$ is often nearly equal to unity. If it is, L becomes nearly equal to (abC_s) as in the previous bridge, and if one knows the coil resistance to be small it is sufficient to compare the proportion of

$$R \text{ to } \frac{1}{C_s\omega}, \text{ i.e. } \frac{\text{Resistance}}{\text{Reactance}}$$

Then there is the same proportion of resistance to reactance in the other arm. This tells the loss in ohms in the coil.

Robinson Frequency Bridge

See Fig. 83 for the circuit of this bridge.

If $a = b$ then

$$\frac{\dfrac{R_1}{jC_1\omega}}{R_1 + \dfrac{1}{jC_1\omega}} = R_2 + \frac{1}{jC_2\omega}$$

$$\frac{R_1}{1 + R_1jC_1\omega} = R_2 + \frac{1}{jC_2\omega}$$

$$jC_2R_1\omega = (1 + jR_1C_1\omega)(1 + jR_2C_2\omega)$$

$$\therefore \ jC_2R_1\omega = 1 - R_1C_1R_2C_2\omega^2 + jR_1C_1\omega + jR_2C_2\omega.$$

Equate reals and the result is quite simple—

$$1 = R_1R_2C_1C_2\omega^2$$

So

$$\omega = \frac{1}{\sqrt{R_1R_2C_1C_2}}$$

This bridge is useful for finding frequency, but when the frequency is known and R_1 and R_2 are also known it finds C_1C_2 as a product. Since the first bridge in this series finds $\frac{C_1}{C_2}$ this gives a valuable means of making absolute measurements in electrical work.

Setting up a Capacity Standard

The use of A.C. at 50 cycles on the grid system of the country to drive clocks has meant that there is a frequency standard of great exactness always available.

The Lissajou figures observed on an oscillograph when a frequency which is a multiple of 50 is put on one axis with the 50 ~ on the other gives a means of calibrating oscillator frequencies exactly. Thus an exact frequency Robinson bridge can be set up (see Fig. 83) and two unknown condensers measured as C_1C_2. The bridge in Fig. 79 measures $C_1 \div C_2$, and so the two condensers are known.

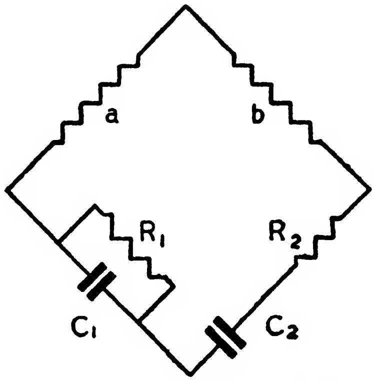


FIG. 83. ROBINSON FREQUENCY BRIDGE

The Hybrid Coil Bridge

The hybrid coil (see Fig. 84) is very useful as a bridge for the more exact determination of coil losses. Its exact function is illustrated in Fig. 85, where it forms part of a two-way amplifier or telephone repeater circuit.

When energy is received from a valve *A* and fed to a coil, this circulates a loop current, but does not energize the input to the valve *B* if the network really balances the line, that is, has the same impedance at all frequencies.

As a bridge (Fig. 84), the coil whose resistance is desired may be tuned with a condenser *C*, so that the result is a pure resistance. This is balanced against *R*. The advantage is low voltages in the bridge arms and stray capacities not giving trouble.

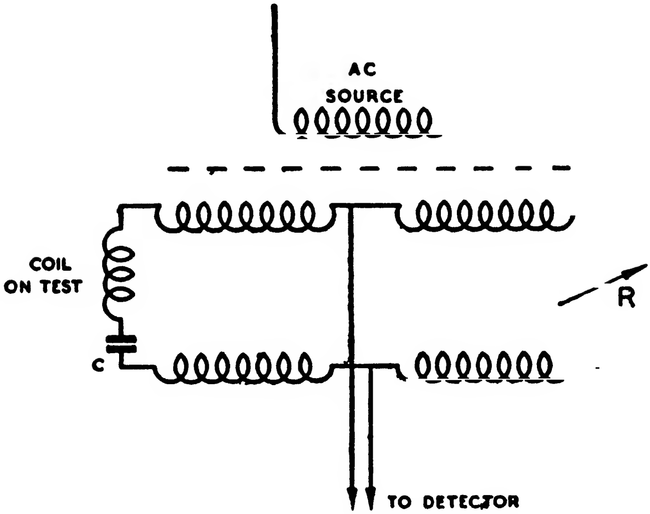


FIG. 84. HYBRID COIL BRIDGE

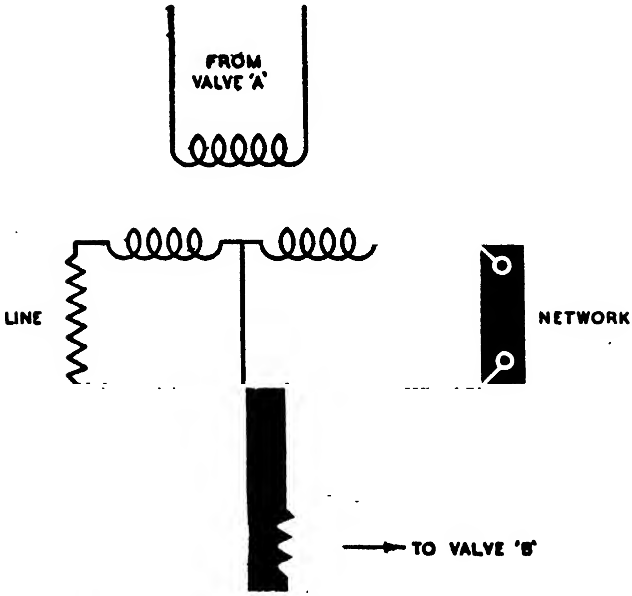


FIG. 85. HYBRID COIL IN TWO-WAY TELEPHONE REPEATER

The Measurements of Phases in Four-terminal Networks

The easiest way to do this is to go all round the four sides of the network with a valve voltmeter and use a generalized Kirchhoff's Law. This may be stated as follows—

The alternating e.m.f.'s in all the branches of any closed circuit form a closed polygon like the polygon of forces. In the case of a four-pole network which is symmetrical, the two voltages across

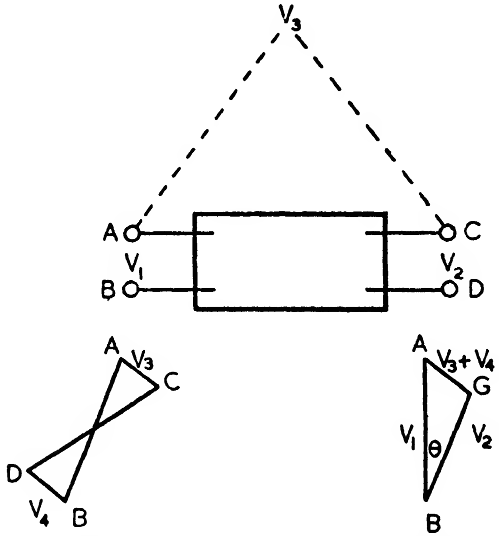


FIG. 86. THE USE OF A VOLTMETER TO MEASURE PHASE CHANGE IN PHASE NETWORKS

the terminals AC and across BD in Fig. 86 are equal and have the same phase relation to the input and output voltages. In other words, the first two are in line and the polygon becomes a triangle of voltages.

To draw the polygon of voltages, draw BA equal on some scale to the voltage between A and B, the input voltage measured with a valve voltmeter. Then draw AC, the voltage between A and C, with BD parallel and equal to it. Then draw CD equal to the output voltage. The points C and D are found by the intersection of arcs, and show the angle required, the angle between input and output voltages. It is given by the formula

$$a^2 = b^2 + c^2 - 2bc \cos \theta$$

or
$$\cos\theta = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\theta = \frac{AB^2 + BG^2 - (2AG)^2}{2AB\,.\,BG}$$

The value of θ calculated in this way is not quite the B of the propagation constant because the measurement is not made with

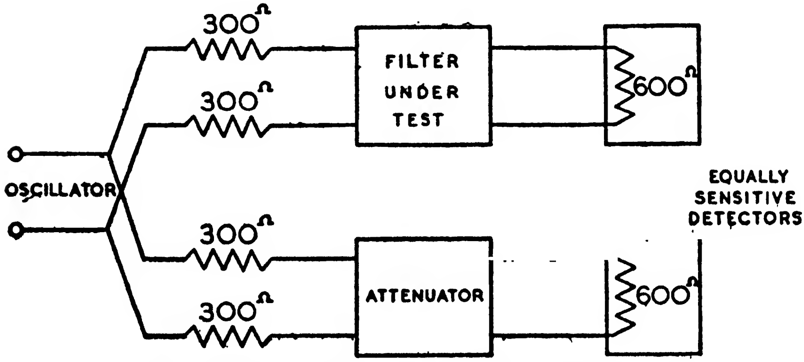


FIG. 87. CIRCUIT FOR MEASUREMENT OF "INSERTION LOSS"

an infinite chain of sections, but is made with one section closed with 600^{Ω}.

This is the angle of phase change of the network when it is "inserted" between 600^{Ω} ends, and is an interesting way of observing the phase changes which are calculated and produced in such networks.

The ratio $\dfrac{BG}{AB}$ is the insertion loss; indeed, this is what the amplification meter measures.

Note on the Measurement of Attenuation of Filters

It is easier to measure a filter between resistance terminations. This gives, not the attenuation proper, but a slightly different result called the "insertion loss," which takes into account any reflections between the filter and the resistance terminations—which is the usual practical case.

It is not obvious that the circuit of Fig. 87 does effectively measure this insertion loss as between 600-ohm resistances. The proof that it does so is to consider that it obviously does it when the oscillator has no impedance and the attenuator is set to make the two detectors read the same. If, due to internal impedance, the oscillator alters, the alteration is the same for both the filter and the attenuator sending voltage. Thus the measurement, which is a comparison of the attenuator with the filter, is not affected.

CHAPTER IX

GENERAL THEOREMS

THE general theorems are as follows—

I. Thévénin's Theorem.
II. Maximum Power.
III. Pure Resistance Networks.
IV. The Filter Theorem.
V. Campbell's Theorem.
VI. Impedance Multiplication.
VII. Foster's Theorem.
VIII. Impedance Frequency Theorem.
IX. "Change of Tempo" Theorem.
X. Resistance in Reactive Networks.
XI. Impedance Inversion.
XII. Frequency Inversion.
XIII. No Loss Networks.
XIV. Generalized Kirchhoff's Law.
XV. Generalized Resistance Formulas.
XVI. Reciprocal Theorem.
XVII. Linearity Theorem.
XVIII. Simplicity of Sine Waves.

Some of these are obvious enough when explained, while others require proof. Their use saves an enormous amount of labour, particularly in alternating current problems. The majority of the theorems do not relate specially to four-terminal networks; they apply to any circuit. Theorems IV, V, XII, and XIII, however, apply particularly to four-terminal networks.

In the pages that follow each theorem is first stated broadly, then particular cases are outlined, after which proof is given and also an example where possible.

Theorem I: Thévénin's Theorem

"The current entering any circuit, due to a voltage in another circuit connected to it, is given by

$$\frac{E}{Z_S + Z_R}$$

where E is the open circuit voltage across the supply terminals A and B in the supply circuit" (see Figs. 88 and 88A).

"Z_R is the impedance looking into the load circuit and Z_S the impedance looking back into the supply circuit, when that circuit is dead." This applies to D.C., A.C., and to transients.

Proof

The following proof makes the theorem self-evident. Suppose the two circuits joined up through an alternator of zero impedance,

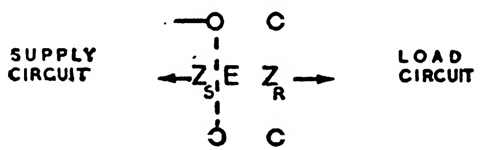


FIG. 88. THÉVÉNIN'S THEOREM (THE PROBLEM)

and suppose the alternator is set to oppose the actual current which it is desired to find, making it zero, the supply circuit being alive too.

The alternator must generate a current which is equal to and

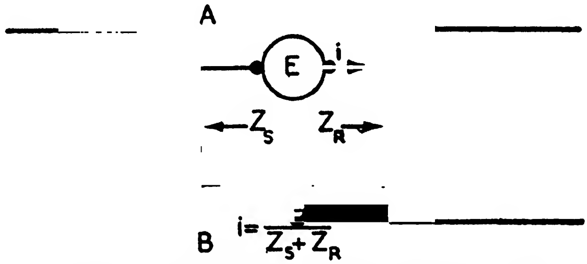


FIG. 88A. THÉVÉNIN'S THEOREM (THE SOLUTION)

opposite to the actual current. If the alternator voltage is then E, the reverse current circulated by it is obviously

$$\frac{E}{Z_S + Z_R}$$

as shown in Fig. 88A. But as the total current in the circuit is now zero, the open circuit voltage in the supply circuit called E must now appear across the terminals A and B even though they are connected to an alternator and a load. This voltage is being opposed by the alternator, so that the alternator voltage is also correctly described as being E, the open circuit voltage across A and B.

What has been done here is to look at the circuit containing the alternator in two ways. First the supply circuit can send a current

i into the load and so, too, can the alternator in the opposite direction, causing zero current as a result, or, alternatively, the alternator can be viewed as opposing the voltage generated by the source or rather the voltage appearing on the terminals A and B. The actual voltage itself in the source network might be larger than the open circuit voltage across AB because it may be an oscillator at the other end of a long line, and AB the end terminals of the line in the left of the diagram.

EXAMPLE

Suppose an oscillator has an open circuit voltage of 50^v and an internal impedance of 1600^Ω pure resistance. What current will be delivered to a pure resistance load of 600^Ω?

The result is

$$\frac{50}{2200} = 22{\cdot}7 \text{ milliamps}$$

In any case where, as is more usual, Z_S and Z_R are partly reactive, the problem becomes one for the j notation, which notation is covered by Theorem XVIII. Thévénin's Theorem leads to another by asking the question, under what circumstances of load impedance will maximum volt-amperes be delivered by a given source of voltage and current?

Theorem II: Maximum Volt-Ampere Theorem

This theorem is of fundamental importance: "Maximum volt-amperes will be delivered to a circuit whose impedance is equal to that of the source, i.e. looking back into the source of current." In the case where the source has an angle the "sink" should have an equal and as nearly an opposite angle as possible.

Proof

Let the source have an impedance Z_S and the circuit supplied Z_R, which may be called x. The current in x is now $\dfrac{E}{Z_S + x}$ by the previous theorem, and the volt-amperes on x will be

$$\frac{xE}{Z_S + x} \times \frac{E}{Z_S + x}, \text{ i.e. } \frac{xE^2}{(Z_S + x)^2}$$

In other words, the reciprocal of this, $\dfrac{(Z_S + x)^2}{xE}$, should be a minimum.

Neglect the E as a constant factor and divide out the x. Then

$$\frac{Z_S^2}{x} + 2Z_S + x$$

should be a minimum.

Differentiation gives $\frac{-Z_S^2}{x^2} + 1 = 0$, or $x = +Z_S$.

What is indicated here is that the apparatus x should have the same impedance as the source, but the actual value of the current and therefore of volt-amperes now x is fixed will be further increased as the angle between the vectors Z_S and Z_R is made greater.

Example

Suppose a long transmission line has a characteristic impedance 400Ω and negative 45° angle. What must the load be for maximum volt-amperes?

In size, Z_R must equal Z_S, so it will be 400Ω. The vector which is most in phase opposition to the line will be an inductance of as nearly 90° as possible.

When pure resistances are chosen, no angle comes in, volt-amperes become watts, and $Z_S = Z_R$ pure ohms. This is the basis of matching two pieces of apparatus by a transformer.

Theorem III: Pure Resistance Networks with Four Terminals

"Four-terminal networks composed of pure resistances cause attenuation; they have the same attenuation to all frequencies. They do not cause phase change. They have a characteristic impedance which is a pure constant resistance, independent of frequency."

Example

The ladder network in Fig. 2 will attenuate or weaken any current applied at S. If there is 1 ampere at A, there will be a fraction of an ampere at B, and a fraction of a fraction at C, and so on. The phase, however, in the case of alternating currents will be the same at A, B, and C.

Corollary to Theorem III

"If every resistance in a pure resistance network is doubled, the characteristic impedance will be doubled, and so on. The attenuation constant remains the same."

Example

A change from 10Ω to 20Ω and from 40Ω to 80Ω in Fig. 1 will double the characteristic impedance, but the attenuation will be unchanged.

Theorem IV: The Filter Theorem

"Networks built up of pure reactances (coils and condensers) act as filters when used with two input and two output terminals."

Proof

The proof consists in considering the formula for the impedance of a length of line short-circuited and then open-circuited at the far end. In the one case

$$Z_s = Z_0 \tanh P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the other

$$Z \text{ open} = \coth P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here P is the attenuation constant of the network; it is the ***natural logarithm*** of current ratio, and Z_0 is characteristic impedance or impedance of an infinite number of sections connected end to end. With a given circuit the j notation would give a result for Z open and another for Z closed. With pure resistances, these could only be reactive or else infinity or zero. They are both j's, either plus or minus in sign, but may differ in sign. Dividing (1) by (2) above,

$$\tanh {}^2P = \frac{Z_s}{Z \text{ open}}$$

If now Z_s and Z open have the same sign the j's cancel; $\tanh {}^2P =$ real $+$ *ve*, then $\tanh P$ is real, P is real, and there *is* attenuation. If, however, at another frequency the two should differ in sign $\tanh {}^2P$ would be negative, $\tanh P$ unreal, and P then imaginary. This means no attenuation but phase change.

The best method of proof to the formulas $Z_0 \tanh Pl$ for a short-circuited line and $Z_0 \coth Pl$ for an open-circuited line consists in studying the reflected waves from the open or short-circuited end. The first reflection is shown for the open circuit in Fig. 89.

Theorem V: Campbell's Theorem

"Filters have a real characteristic impedance, i.e. a pure resistance in their pass bands, and a j or fully reactive characteristic impedance in their attenuation bands."

Proof

The two equations (1) and (2) above must be multiplied, giving $Z_0^2 = Z_s Z_{\text{open}}$. Suppose Z_s and Z_{open} have opposite signs, which is the condition for a pass band. As they are both j's, Z_0 comes to

be real and positive. Then Z_0 is real, i.e. pure resistance. If, however, as in an attenuation band Z_s and Z_{open} have the same sign to the j, then Z_0^2 = *a negative real*, and Z_0 is imaginary, which means a pure reactance. For this reason a further theorem is possible.

Corollary to Theorem V

"In order to study a particular filter circuit it is enough to study Z_0 first of all."

This will indicate the pass bands; formulas for sizes of coils and

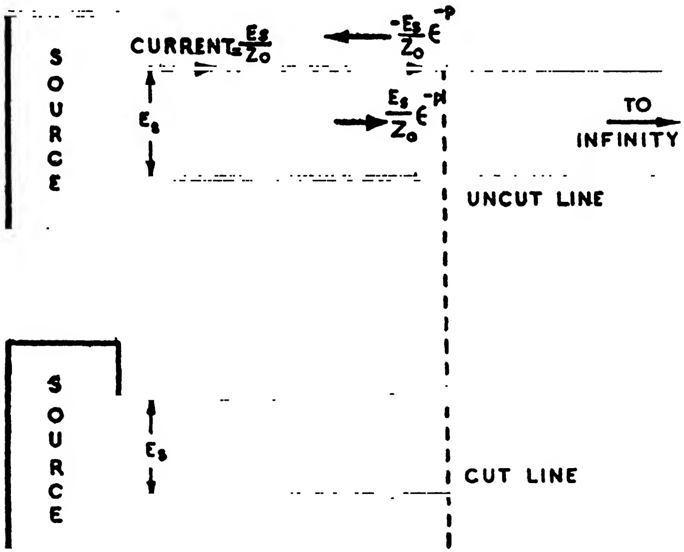


FIG. 89. REFLECTION OF CURRENT ON OPEN-CIRCUITED LINE

condensers may be worked out without troubling about the formula for attenuation. That can follow later.

Examples of this are the derivation of the formulas in the text for low pass, high and band pass filters.

Theorem VI: The Impedance Multiplier Theorem

"The impedance of a circuit containing only pure reactances will be doubled, at any and every frequency, if the coil inductances are each doubled and if the condensers are each halved in capacitance. Impedances may be so increased or diminished by any factor."

Proof

In the j notation, the impedance of a coil is $jL\omega$ and therefore proportional to L. That of a condenser is $\frac{1}{jC\omega}$ and so inversely proportional to C.

Any amount of putting such in series or parallel if each is altered by the same factor yields a result which is altered by that factor.

EXAMPLE

A coil of 0·1 henry and a condenser of 10μF are in parallel. What will give three times the impedance?

Answer: A parallel coil and condenser of 0·3 henry and $3{\cdot}33\mu$F respectively.

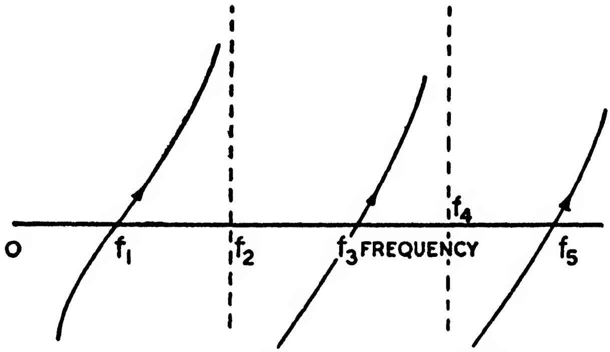


FIG. 90. GRAPHICAL FORM OF THEOREM VII

Theorem VII: Foster's Theorem for Reactances

"Any network of reactances alone must present a value at any frequency which rises in value if already positive (tends to 0 if negative) as the frequency is raised. Zero values alternate with infinite ones and the infinite ones are approached towards *plus* infinity."

That is the theorem in words. It has a graphical expression which is simpler still (see Fig. 90). This shows that the curves *always* move upwards as the frequency is increased.

Mathematically the theorem can be expressed as follows—

$$Z = \frac{-jH\,(\omega_1{}^2 - \omega^2)\,(\omega_3{}^2 - \omega^2)\,(\omega_5{}^2 - \omega^2)}{(\omega_2{}^2 - \omega^2)\,(\omega_4{}^2 - \omega^2)}$$

Theorem VIII: Impedance Frequency Theorem

"If a circuit built of reactances only is desired to have the same impedance to all frequencies one octave higher or lower, and so on,

each coil and condenser (inductance and capacitance) must be halved or doubled, and so on."

Proof

This follows because C and L are *both* multiplied by ω in the j notation.

EXAMPLE

A coil and condenser of 0·2 henry and $4\mu F$ in series has a certain impedance characteristic with frequency. What circuit would have a similar frequency curve at one-fifth the frequency scale, i.e. the same at 160~ as the above circuit has at 800~, and so on?

Answer: 1 henry in series with $20\mu F$.

Application

This theorem enables filters with any desired *impedance* to be designed from one of, say, 600^{Ω} impedance.

Theorem IX: The " Change of Tempo " Theorem

"Circuits containing pure reactances which behave in any given way as regards battery or alternating or other e.m.f.'s will behave in a precisely similar way when all coil inductances and condenser sizes are reduced, provided the tempo of the applied e.m.f. is increased in the same ratio. The tempo of the currents flowing will be speeded up too."

Proof

This theorem follows for the same reason as the last, and because of Fourier's series and other work. That is to say, the theorem is true even if the voltages are not sine waves. If, however, the applied e.m.f. is Heaviside's unit function, i.e. a 1^v battery switched on and left on, no change of tempo is possible.

EXAMPLE

Suppose 200^v 50 c/s causes a current of 10 amps. in a certain coil in a certain circuit. If all coils and condensers are halved, the coil will now carry 10 amps. still if the frequency is put up to 100 c/s, the voltage remaining the same.

In the first case it was 10 amps. at 50 c/s, now it is at 100 c/s.

Application

Filters for a higher or lower range of frequency may be designed when a filter for one particular cut-off has been designed. Care is needed with band filters, because each cut-off is doubled or halved and so the band width in cycles is doubled or halved.

Theorem X: Resistance in Reactive Networks

"Any circuit derived from another using only a frequency change keeps all resistances as they were, but an impedance change only requires changed values of resistance."

In Theorem VI, in doubling the impedance one doubles all resistances, but in Theorem VIII they must be made the same at the new frequency, which may mean a different quality of coil.

Theorem IX holds if all resistances are made the same at the new frequency as the old ones at the old frequency.

Theorem XI is modified, and in working out examples complex quantities must be used.

In general, in filter design, condenser and coil losses should be kept low. The reactances make the filter.

Theorem XI: Impedance Inversion

"Any impedance containing reactances and resistances can be used to produce an 'inverted impedance' such that the product of the two at any and therefore at every frequency is a constant resistance, say R^2."

Proof

This follows from the nature of the process and from putting coils and condensers in series and parallel by the j notation.

Where two components in the first circuit are in series the corresponding ones in the inverted circuit must be put in parallel.

Where two components (or portions of circuits) are in parallel, the corresponding portions in the inverted circuit must be put in series. A coil L in the first circuit becomes a condenser C' in the inverted circuit such that

$$\frac{1}{jC'\omega} = \frac{R^2}{jL\omega} \text{ or } \frac{L}{C'} = R^2.$$

A condenser C in the first circuit becomes a coil L' in the second, and $\frac{L'}{C}$ again equals R^2. A resistance r in the first circuit becomes $\frac{R^2}{r}$ in the second.

EXAMPLE

The arrangement is shown in Fig. 91.

Application

For making networks where the characteristic impedance is desired to be a pure resistance at all frequencies. Also a series-derived filter by inversion becomes shunt-derived.

Theorem XII: Frequency Inversion

"If a network of reactances is replaced by a new network of the same pattern (i.e. the same geometrical circuit, say a ladder) and each component is replaced by the one which would resonate with it at a frequency f_x, then the second circuit at any frequency, say $3f_x$, a multiple of f_x, behaves as the first circuit would at a frequency $\frac{f_x}{3}$, as regards attenuation and as regards the size of its impedance, but the impedance angle at any particular frequency will be reversed."

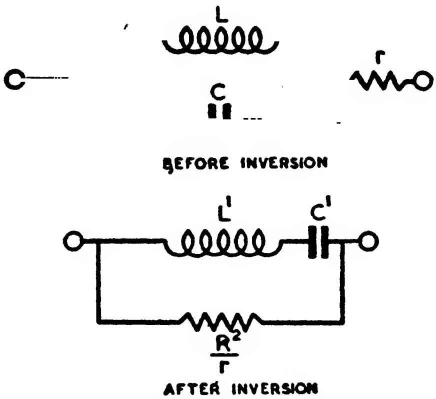


FIG. 91. INVERTED IMPEDANCES

Proof

First, the impedance of, say, a coil L at one-third of the frequency ω_x, which is $\frac{\omega_x}{3}$, is $\frac{jL\omega}{3}$ . . . . . (1)

Note this.

The corresponding condenser C makes $\frac{1}{LC} = \omega_x^2$, so $C = \frac{1}{L\omega_x^2}$.

Its impedance is $\frac{1}{3jC\omega_x}$ at $3\omega_x$, which comes to $\frac{L\omega_x^2}{3j\omega_x} = \frac{L\omega_x}{3j}$. This is the same as (1) above, except that it is $\frac{1}{j}$ or $-j$ instead of j.

The figure 3 has been taken at random just to avoid too many symbols. It can be any number.

EXAMPLE

A coil of 0·1 henry in a low pass filter whose cut-off is 1000 c/s may be inverted *about* the cut-off frequency, giving a high pass filter with a condenser in that place of value

$$0{\cdot}1 \times 2\pi \times 1000 = \frac{1}{C2000\pi}$$

Solve this for C, and the required condenser is known.

Application

A low pass filter on inversion gives a high pass filter; a high pass, a low pass; and a band pass a band elimination.

Theorem XIII: Networks with No Attenuation

"If a network with no attenuation, which must therefore be built of reactances, is placed between resistance terminations (equal ones for simplicity), it may cause a loss or it may not. It will cause no loss if its characteristic impedance Z_0 is exactly equal to the terminal resistances even if there is phase change in the network. It will cause no loss if there is no phase change in the network, even if its Z_0 does not match the terminal resistance. It will cause a loss if there is both mismatching *and* internal phase change."

Proof

This follows from an examination of the telegraph equation, giving the received current at the end of a line with apparatus Z_S at the sending end and Z_R at the far end. The line has characteristic impedance Z_0 and propagation constant P. The reader is referred to books on lines.

It is

$$i_r = \frac{E}{(Z_S + Z_R)\cosh P + \left(Z_0 + \frac{Z_S Z_R}{Z_0}\right)\sinh P}$$

If now the cable is taken away the result is

$$\frac{E}{Z_S + Z_R}$$

by a simple circuit. Here E is the sending end voltage. The division of these gives the current ratio reduction caused by the insertion of the filter. We are here calling the filter a line with Z_0 and P. The current ratio turned to népers or decibels is called the "insertion loss." It is what the "amplification meter" measures. If $Z_S = Z_R = R$ a resistance, this is

$$\text{Cosh } P + \left(\frac{Z_0}{2R} + \frac{R}{2Z_0}\right)\sinh P$$

This is an exceedingly useful formula. Here $\frac{R}{Z_0}$ is a mismatching factor because we are in a pass band for this theorem and Z_0 is a pure resistance like R. Call $\frac{Z_0}{R} = \phi$.

As P is unreal, call it jB. Loss ratio $= \cos B + j\left(\frac{\phi}{2} + \frac{1}{2\phi}\right)\sin B$.

Here $\frac{\phi}{2} + \frac{1}{2\phi}$ is the average of ϕ and its reciprocal.

To prove the theorem, if $B = 0$, $\sin B = 0$, $\cos B = 1$. Therefore the current ratio is unity.

If now ϕ is unity, the current ratio is $\cos B + j \sin B$, which is a unit vector.

Theorem XIV: Generalized Kirchhoff's Law

"(1) The alternating e.m.f.'s in the separate branches of any closed circuit, expressed as vectors, form a closed polygon, like the polygon of forces. That is to say, the vector sum is zero.

"(2) The vector sum of the alternating currents going to any one point is also zero."

The result of this is that any formula for a circuit, derived for pure resistances, can be used to obtain results with reactances, using the j notation, and is equally valid for resistances together with reactances using complex quantities. For example, the formula for the impedance of a "T" attenuator can be used to find the characteristic impedance of the low pass filter.

This is really an example of a further theorem.

Theorem XV: Resistance Results Generalized

Circuit formulas for Z_0 and P worked out for resistances may be used at once for A.C. work if reactances are put in instead of, or as well as, the resistances.

Theorem XVI: The Reciprocal Theorem

"If in any circuit not containing one-way paths such as one-way amplifiers, a voltage E at a place called A causes a current I at a place called B then if the voltage E were put in at B it would cause the current I at A."

It is a remarkable theorem. (See Fig. 92.)

FIG. 92. THE CURRENT IS THE SAME IF BATTERY AND AMMETER ARE INTERCHANGED WHATEVER THE CIRCUIT

Theorem XVII: The Linearity Theorem

"Good reproduction of waves is not in general possible, and the j notation is invalid, if an increase in voltage does not cause a strictly

proportional increase in current in each part of the circuit. In other words, the inductances and capacities must remain the same: the constants must *be* constant."

This replies to receivers. In a crystal telephone, for instance, the motion of the crystal ought to be proportional to the voltage on it. In practice it is, over a wide range of voltages.

Theorem XVIII: Simplicity of Alternating Currents

"If the circuit constants are really constant, because a sine wave, differentiated, still has the same shape and frequency, and, further, because two sine waves of the same frequency always add up to make another of the same frequency, it follows that a *sine wave* e.m.f. will cause a *sine wave* current of the *same* frequency however much the circuit is complicated by the inductances with their introduction of the differential and by condensers with their introduction of the ntegral. This makes the *j* notation possible."

It is a general law of nature on which, for example, the theory of sound rests.

CHAPTER X

CAUER FILTERS

In the last few years filter design has been approached from a rather new angle. The American inventors who made the first filters (of the ladder type) took certain circuits and made a complete study of their possibilities. They developed a design technique over a period of some years. The emphasis was, however, still on the circuits. The engineer building a carrier system, for example, said, "We need such

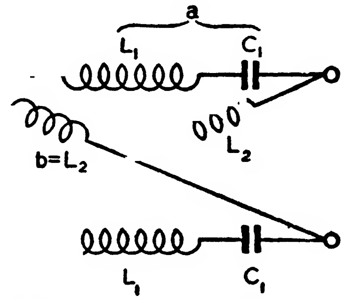


Fig. 93. Simple Lattice Low Pass Filter

and such a loss at such and such a frequency: take so many sections of such a section or use different sections each with its own circuit and see if it is enough when the sections are all put together." If the estimated filter was not good enough, a bit could be added.

Cauer, on the other hand, started with the mathematical functions. Since the lattice circuit has the simplest formula for its attenuation (and for its characteristic impedance, too) he worked on lattice circuits. Then if a and b are the arms in Fig. 93 the attenuation is given by the lattice attenuation formula

$$\text{Tanh}\,\frac{P}{2} = \sqrt{\frac{a}{b}}$$

an easy function.

Thus the function to be considered is $\sqrt{\frac{a}{b}}$ where a is $jL_1\omega + \frac{1}{jC_1\omega}$ if the arm is a coil and condenser in series, but different if they are in parallel, and more complicated if the circuit is more complicated.

Cauer has worked out a method of finding how many coils and condensers are needed, that is, the number of each and their sizes, from the attenuation and impedance requirements. This is the proper way, and is a reversal of what is usually done.

We normally draw a filter of so many sections, calculate its attenuation, and if it is not enough put another section on and recalculate the attenuation. As distinct from this method, Cauer shows tables for 1, 2, 3, etc., coils and so many condensers. Each filter is given a "class" number for its attenuation. A low pass filter of Class 3 can be built of three coils, so the tables show what can be done with such and such a number of coils.

He did not neglect the characteristic impedance, but showed how to make it level in the pass band, within such and such limits. He considered originally lattice filters because of the simple formulas for the characteristic impedance and attenuation of the lattice. $\sqrt{\frac{a}{b}}$ is simpler than $1 + \frac{a}{2b}$ because of the extra 1.

Anyone who has tried to make a filter with a new circuit pattern knows that one soon gets long formulas containing powers of ω, the circular frequency.

Cauer's work has been little understood for the reason that he made three simultaneous advances. These were—

(1) The use of a new symbol Ω (not to be confused with the same sign for ohms) which means

$$\frac{\text{Frequency}}{\text{Cut-off frequency}}$$

for a low pass filter. It is a reciprocal in the case of a high pass filter. (We do this in drawing graphs and make the graph do for any cut-off frequency by writing $\frac{f}{f_0}$ for a low pass filter graph.)

(2) The use of trigonometrical tables to work out the frequent expressions of the form $\sqrt{1 - x^2}$. This is a good dodge in calculation.

(3) The use of Tschebbytcheff's mathematics to get the best result out of a given complexity of algebra, i.e. a given complexity of circuit or number of coils and condensers.

The first advance, the use of capital Ω, is one affecting the algebra, and is closely bound up, but not necessarily so, with the polynomial functions of Tschebbytcheff. It is a question of using a generalized frequency Ω rather than the actual frequency in cycles per second. The Ω is a function of the ω, which is $2\pi f$.

This is common to all filter design since a low pass filter, say, of cut-off 1000 cycles and worked at 2000 cycles will exactly compare with a similar filter of cut-off 10,000 cycles worked at 20,000 cycles, and so on. One commonly uses $\frac{\omega}{\omega_0}$ or $\frac{f}{f_0}$ on graphs of filter performance, as it is the ratio of frequency/cut-off frequency which counts in low and high pass filters. Cauer has extended this to band pass

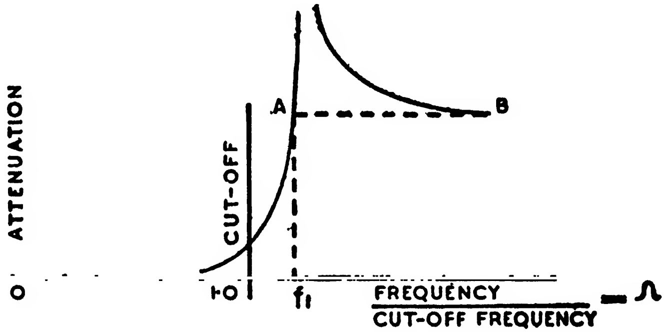


FIG. 94. THE ATTENUATION CURVE OF THE SIMPLE LOW PASS LATTICE AND ITS EQUIVALENT LADDER CIRCUIT

filters. When $\Omega = 0$ the working frequency denoted is the mean frequency $\sqrt{f_1 f_2}$ musically in the middle of the pass band. $\Omega = 1$ is the upper cut-off frequency f_2, and $\Omega = -1$ is the lower end of the pass band.

This is somewhat unusual and makes Cauer's curves difficult to understand at first glance. The Ω is a ratio and is simply the *musical interval* between the frequency and the cut-off frequency for low and high pass filters.

Cauer's great advance, however, was to calculate the circuit components from the desired attenuation. Suppose the desired attenuation is 50 db in the case of a one kilocycle low pass filter. No one can make a filter with 50 db at the cut-off, because at that point the attenuation is zero. There must then be a little space. One can get 50 db at and above a frequency of 1200 c/s, say. The limit of 1200 can be tightened up by using more coils or else by reducing the 50 db to 40 db.

Look for a moment at the curve in Fig. 94 for a simple *m*-type ladder filter. Make *m* small and the peak of attenuation moves nearer to the cut-off. Make *m* large and it moves away to the right merely by a change of *sizes* of components. Unhappily the nearer

to the cut-off the peak is made, i.e. the steeper the attenuation curve just after cut-off, the lower the value of attenuation to which the curve falls, at high frequencies. Draw a horizontal line at the level to which the curve eventually falls and produce this backwards to cut the rising attenuation curve at A. This point A is a definite and valuable point. It is a figure of attenuation *at and above an associated*

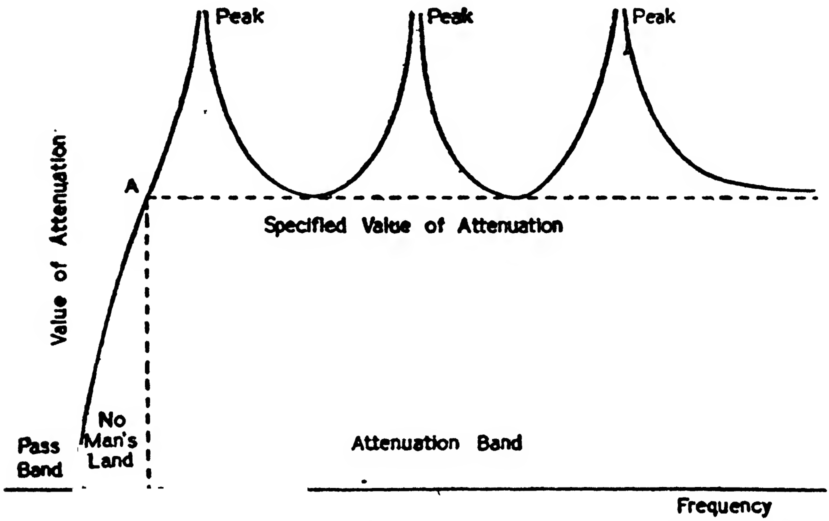


FIG. 95. AN ATTENUATION CURVE WITH A CORRECT SET OF VALUES OF THE COEFFICIENTS IN THE Q FUNCTION FOR THE ATTENUATION, GIVING THE TROUGHS ALL AT THE SAME LEVEL AND LEVEL WITH THE FINAL VALUE OF ATTENUATION

figure of frequency. The association is quite definite, but depends on *two things*, on the circuit and on the value of m used. Here m is a parameter.

The drawing Fig. 94 contains a frequency f_1, which is the *lowest useful frequency* of this filter. It is the lowest useful frequency because the attenuation is above the height A at all frequencies above this f_1. We have proved in an early chapter that attenuation curves are always of zero attenuation at the cut-off frequency. They take a number of cycles to climb up to the useful value. Thus if one asks, "What attenuation can be secured by one, two, or three sections of, say, an m-type ladder low pass filter?" then the request *must* contain information with regard to the interval which may separate

the cut-off frequency from the lowest useful frequency. The values of the *m*'s control the initial steepness of the curve and its final drop too.

Consider next a three-section *m*-type filter with the three sections having their peaks spaced in such a manner that the troughs come level with each other and level with the line drawn horizontally through the level of attenuation at infinity. (See Fig. 95.)

With three sections one can make the curve steeper at first in its initial rise, but if this is done the troughs will come lower. So with correct design the troughs are all as shown in Fig. 95, but there is still a certain latitude. Corresponding to any *specified value of attenuation* there is a given value of lowest useful frequency.

Thus tables are needed, one table for a one-section filter, another for a two-section filter, and so on, and each table must show the value of attenuation for various values of "no-man's land," the (musical) interval between the cut-off frequency and the lowest useful frequency.

This matter of a lowest useful frequency may seem hard to grasp. Begin with a simple low pass ladder filter. Its attenuation curve is given by $\cosh P = 1 - 2\Omega^2$, the Ω being

$$\frac{\text{Frequency}}{\text{Cut-off frequency}}$$

which is a ratio or frequency interval in the musical sense.

The simple filter is called Class 1′. When $P = 2$ népers, for example, $\cosh P = 3{\cdot}76$. Thus $2\Omega^2 - 1 = 3{\cdot}76$

$$\Omega = 1{\cdot}52$$

(See the table given for Class 1′ filter, below.)

TABLE 19

CAUER'S CLASS 1′ FILTER

P	Cosh P	$2\Omega^2$	Ω^2	Ω	k
2	3·76	4·76	2·38	1·52	0·658
3	10·068	11·068	5·534	2·35	0·425
4	27·308	28·308	14·154	3·76	0·266
5	74·21	75·21	37·60	6·12	0·164
6	100·6	101·6	50·8	7·1	0·141

In Fig. 14 one cannot use the filter at a frequency below 1·52 times the cut-off frequency if one desires an attenuation of two népers. This makes a reciprocal of 0·658, which is the *k* to be used when studying impedance variations.

Thus, with a single section of a plain low pass filter (not a derived section) one can get two népers at any frequency above 1·52 times the cut-off value. Let us see what can be done with a derived filter. To do this it is only necessary to put the frequency of infinite attenuation in such a place that the attenuation in Fig. 94 finally falls to two népers. Draw a line horizontally at a height of two népers, and notice where it cuts the rising curve, i.e. find the frequency for the point A.

The calculation is best done by starting with the mathematical functions for attenuation. The lattice has the easiest functions for $\tanh \tfrac{1}{2}P = \sqrt{\dfrac{a}{b}}$ and $Z_0 = \sqrt{ab}$. Notice that Fig. 93 shows the lattice circuit equivalent to the derived type of ladder filter. It has been shown that a ladder of several sections of m-derived type can be replaced by a complicated lattice. This brings us to the study of lattice filters and the possibilities which are contained in a function $\sqrt{\dfrac{a}{b}}$, which is a function of frequency. The more complicated the filter is, the more complicated $\sqrt{\dfrac{a}{b}}$ is.

If, as in Fig. 93, the arm a is a simple coil and condenser in series, and if the other arm is simple too, then the expression for $\sqrt{\dfrac{a}{b}}$ may be as simple as $\dfrac{\Omega}{\sqrt{\Omega^2 - 1}}$, or at least if not as simple as that it may be $\dfrac{\Omega}{H\sqrt{\Omega^2 - 1}}$ where H is a fraction, a number, and Ω means frequency or some number representing it.

The reason for this form of expression may be seen if the circuit in Fig. 93 is considered.

Here a is $jL_1\omega + \dfrac{1}{jC_1\omega}$ and b is $jL_2\omega$. The result is that $\sqrt{\dfrac{a}{b}}$ becomes

$$\sqrt{\frac{jL_2\omega}{jL_1\omega + \dfrac{1}{jC_1\omega}}}$$

and this is

$$\sqrt{\frac{-L_2C_1\omega^2}{1 - L_1C_1\omega_2}} = \sqrt{\frac{L_2}{L_1}}\sqrt{\frac{\omega^2}{\dfrac{1}{L_1C_1} - \omega^2}}$$

which is $$\frac{1}{H}\frac{\omega}{\sqrt{\omega^2-\omega_0{}^2}}$$

if H is called $\sqrt{\frac{L_1}{L_2}}$.

It is convenient to gather up the odd constants into H and also to express $\frac{1}{L_1C_1}$ as ω_0 because this is the cut-off frequency. It is recognized as the cut-off frequency because $\sqrt{\omega^2-\omega_0{}^2}$ is real for high values of ω, i.e. values greater than ω_0, so tanh $\frac{1}{2}P$ is real. This makes P real because of the property of tanh x that it is real when x is real. When, however, ω is less than ω_0 the result emerges that P is unreal and that means a phase shift, but no attenuation. Attenuation above $\omega - \omega_0$ but none below means ω_0 is the cut-off, and so it is a low pass filter.

Since it is the ratio of ω to ω_0 which controls the value of the function it is convenient to divide top and bottom by ω_0 and call $\frac{\omega}{\omega_0}$ a new variable Ω. This is one of the things that makes Cauer's work look hard. The formula $H\frac{\Omega}{\sqrt{\Omega^2-1}}$ does not look intelligible. Let us discuss this formula.

The Ω is a general expression for frequency. If the cut-off is 800 c/s, a frequency of 400 c/s makes $\Omega = \frac{1}{2}$, and so on. The value $\Omega = 1$ is the cut-off, and the formula must contain $\sqrt{\Omega^2-1}$ in order to go from real to unreal values of attenuation as the frequency Ω is reduced below 1, i.e. past the cut-off frequency.

Further, at high values of frequency the $\sqrt{\frac{a}{b}}$ tends to become constant and in particular tends to take the value H. Thus the attenuation at very high frequency depends on H. At some intermediate frequency between the cut-off $\Omega = 1$ and infinity, the value of attenuation is infinite because the Wheatstone bridge balances. That is, $\frac{H\Omega}{\sqrt{\Omega^2-1}} = 1$.

This brings us to the main feature of Cauer's work. The final attenuation at very high frequency is given by tanh $\frac{1}{2}P = H$.

***H* decides the attenuation below which the curve will not fall after it has gone to the peak at**

$$\frac{H\Omega}{\sqrt{\Omega^2-1}} = 1.$$

If, then, in Fig. 96 the attenuation falls to a final value given by the line *FF′*, the guaranteed attenuation *OF* is only reached at a frequency Ω_L decided by the point *A*, which is the attenuation the filter finally reaches at very high frequency.

Notice that if coils L_1 and L_2 are made nearly alike, then *H* is

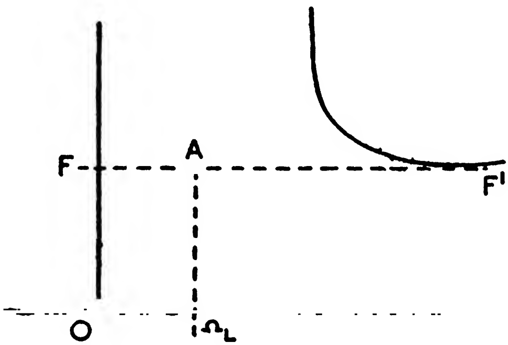


FIG. 96

nearly 1 and the final attenuation is high. The filter looks a good one, *but* the point *A* moves over to the right.

This is the effect of altering the factor *m* to a value near 1 in the *m*-derived type of ladder filter. The lowest useful frequency becomes higher up. Thus, a high attenuation from a simple circuit means that the filter cannot be used just above cut-off. There is a kind of "no-man's land," as previously stated, between the cut-off and the lowest useful frequency. This becomes wider the nearer *H* is made to 1 in the design, which means that at and above (but not below) a value of frequency Ω_L, the guaranteed attenuation is reached. Ω_L may be called, one suggests, **the lowest useful frequency.**

It is better to concentrate, not on the attenuation, but on $\dfrac{\sqrt{\Omega^2 - 1}}{H\Omega}$, which has been called a *Q* function, and which is denoted by *Q*(1) because it is the simplest *Q* function.

The graph of this function is given in Fig. 97.

The expression for the *Q* function may be shown in that way, or it may be given as the reciprocal. In using tanh *x* tables the function must always be a fraction, since tanh *x* is always fractional. In any

case, the reciprocal merely means a phase reversal, electrically equivalent to an interchange of the arms a and b, which does not affect the filter as regards its performance.

It becomes possible then to make up tables of values of Ω_L against values of minimum attenuation. At the same time it is convenient

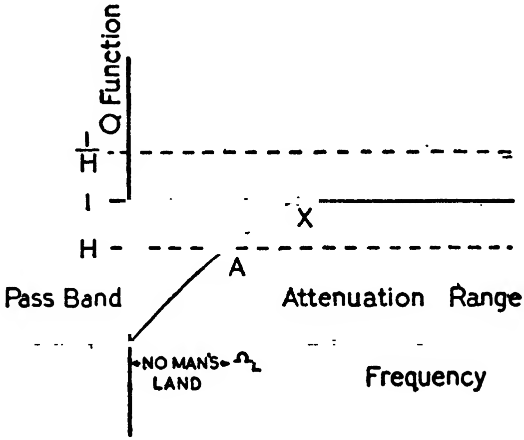


FIG. 97. THE SIMPLEST Q FUNCTION, $Q(1)$

X is the peak attenuation frequency and A the point of lowest useful frequency.

to plot H and the frequency Ω_1 of infinite attenuation. The calculation for this simple circuit and function Q is not difficult. Select H values and read off $\frac{1}{2}P$ from Tables of Hyperbolic Tangents since $\tanh \frac{1}{2}P = H$.

Next make $\dfrac{H\Omega}{\sqrt{\Omega^2 - 1}} = \dfrac{1}{H}$, and this gives the condition for finding the point A for the lowest useful attenuation. The reasoning is that the Q function rises from zero or else falls from infinity to 1, passes through 1 and finishes up at very high values of Ω by tending towards a value H or else $\dfrac{1}{H}$, depending on whether the function is put one way up or the other.

If it is rising towards $\dfrac{1}{H}$ it is this $\dfrac{1}{H}$ and H which represent the final value of the attenuation, called the guaranteed attenuation. It is the attenuation at and above the lowest useful frequency.

The H and $\dfrac{1}{H}$ represent equal unbalance of the bridge. When the Q function reaches H the attenuation is the same as the final

value, so if we put $\frac{\sqrt{\Omega^2 - 1}}{H\Omega} = H$ this gives Ω_L, the lowest useful frequency in terms of H for this simple function $Q(1)$.

Thus $$\frac{\Omega^2 - 1}{H^2\Omega^2} = H^2 \text{ or } \Omega^2 - 1 = H^4\Omega^2$$

so $$\Omega^2(1 - H^4) = 1$$

$$\therefore\ \Omega_L = \frac{1}{\sqrt{1 - H^4}}.$$

This is the relation between the lowest useful frequency and H.

Next, when $Q(1) = 1$ there is infinite attenuation, so

$$\frac{\Omega^2 - 1}{H^2\Omega^2} = 1$$

gives the Ω of infinite attenuation of the filter, i.e. $\Omega^2 - 1 = H^2\Omega^2$

$$\therefore\ \Omega^2(1 - H^2) = 1$$

or $$\Omega_{inf} = \frac{1}{\sqrt{1 - H^2}}.$$

One may say, since H determines the attenuation, why not abandon its use in favour of a given number of népers? The relation is $\tanh \frac{1}{2}P = H$. The reason for keeping H is that the same sort of argument which is used throughout the attenuation band for attenuation is applicable to the impedance in the *pass* band. If it is to be level with frequency, $\sqrt{ab}$ must hug the value 1 in the pass band to get a level impedance just as $\sqrt{\frac{a}{b}}$ must hug unity in the attenuation band.

The Use of Trigonometry for Avoiding Arithmetic

Cauer uses trigonometrical tables to find the value of such expressions as $\frac{1}{\sqrt{1 - H^2}}$. This makes the reading look almost like another language, but it is simple enough. Let $H = \sin\theta$. Then $1 - H^2 = \cos^2\theta$ and $\sqrt{1 - H^2} = \cos\theta$, so its reciprocal is $\sec\theta$. Thus $\frac{1}{\sqrt{1 - H^2}}$ called $\Omega_1 = \sec$ of an angle whose sin is H.

This is easy to find in tables where all the functions are plotted in parallel columns, thus—

θ	Sin	Cos	Tan	Cosec	Sec	Cot
0°	0	1	0	Inf.	1	Inf.
	0·8				1·667	

It is not necessary to look at the angle; look for $H = 0{\cdot}8$ under "Sin" and see the 1·667 under "Sec" on the same line. The accompanying table gives a few of the simpler relations.

The whole technique needs extension to more complicated circuits, Table 20 shows how it is done.

TABLE 20

TABLE OF FUNCTIONS FOR EASY ARITHMETIC

$\sin\theta = \sqrt{1 - \cos^2\theta}$ gives $y = \sqrt{1 - x^2}$
$\operatorname{cosec}\theta = \dfrac{1}{\sqrt{1 - \cos^2\theta}}$ gives $y = \dfrac{1}{\sqrt{1 - x^2}}$
$\cosh^2\theta = 1 + \sinh^2\theta$ gives $y = \sqrt{1 + x^2}$
$\operatorname{sech}\theta = \dfrac{1}{\sqrt{1 + \sinh^2\theta}}$ gives $y = \dfrac{1}{\sqrt{1 + x^2}}$

A more complicated circuit may be made by putting more coils and condensers in the arms of the lattice. The ratio of the arms must give an expression with $\sqrt{\Omega^2 - 1}$, or else we should not go from a pass band to an attenuation band when Ω becomes greater than 1. In other words, it would not be a low pass filter.

The next more complicated expression is found by putting two such filters as the above in line end to end. They should have different peak attenuation frequencies so placed that the peak of one section helps to fill up the long trough of the other, giving an attenuation curve as in Fig. 95 (page 142).

If the filter has an attenuation which finally falls, as there shown, at high frequencies, the bottom of the troughs between the two peaks should lie on that line. If a trough comes below, i.e. below the point A, then A does not give the lowest useful frequency.

First let us show that the Q function for two sections has the form

$$\frac{H\Omega\sqrt{\Omega^2 - 1}}{\Omega^2 - B^2}$$

where H is a value of deviation from 1 for the whole circuit. Using two sections, H may be much closer to 1 for a given width of "no man's land" than for one section, so for a given figure of lowest useful frequency there is a higher attenuation. Let us call the attenuation of the finished filter P. Then $\tanh \frac{1}{2}P =$ the function $Q(2)$.

But $$\tanh \tfrac{1}{2}P_1 = \frac{H_1\Omega}{\sqrt{\Omega^2 - 1}} = \text{the } Q(1) \text{ for first section}$$

and $$\tanh \tfrac{1}{2}P_2 = \frac{H_2\Omega}{\sqrt{\Omega^2 - 1}} = \text{the } Q(1) \text{ for second section}$$

The H_1 and H_2 are for each section taken alone. The attenuations add up, but the trigonometrical formula for the tanh of the total $\frac{1}{2}P$ is as follows—

$$\text{Tanh } \tfrac{1}{2}(P_1 + P_2) = \frac{\tanh \frac{1}{2}P_1 + \tanh \frac{1}{2}P_2}{1 + \tanh \frac{1}{2}P_1 \tanh \frac{1}{2}P_2}$$

This makes $\tanh \frac{1}{2}P$, which is $Q(2)$ to be

$$\frac{\dfrac{H_1\Omega}{\sqrt{\Omega^2 - 1}} + \dfrac{H_2\Omega}{\sqrt{\Omega^2 - 1}}}{1 + \dfrac{H_1H_2\Omega^2}{\Omega^2 - 1}}$$

Simplifying this,

$$\frac{(H_1 + H_2)\Omega\sqrt{\Omega^2 - 1}}{\Omega^2 - 1 + H_1H_2\Omega^2}$$

$$= \frac{(H_1 + H_2)\Omega\sqrt{\Omega^2 - 1}}{\left(\Omega^2 - \dfrac{1}{1 + H_1H_2}\right)(1 + H_1H_2)}$$

Thus $Q(2)$ has the form

$$\frac{H\Omega\sqrt{\Omega^2 - 1}}{(\Omega^2 - B^2)}$$

where one gives values to H as before, calculates the final attenuation from $\tanh \frac{1}{2}P = H$ because the final attenuation is the guaranteed attenuation and one wishes to find B and the two peaks, or rather frequencies, for each of the peak attenuations. The $Q(2)$ function has a curve like that shown in Fig. 98.

It is desired to see that at the maximum value, i.e. the maximum departure from unity which gives the trough of attenuation, the value is $\frac{1}{H}$. To secure this, differentiate $Q(2)$ to find Ω for the turning

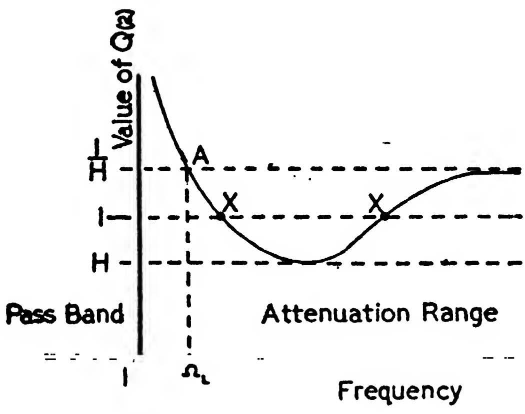


FIG. 98. GRAPH OF FUNCTION $Q(2)$
XX are peak attenuations.

point of the $Q(2)$ curve and, putting it in the function $Q(2)$, equate the result to $\frac{1}{H}$. This finds the constant B when one has chosen H.

Equating the curve itself to H should give the value Ω_L for the point A, that is, when one has found the constant B.

To find Ω for the trough of attenuation, it is convenient to differentiate

$$Q(2) = \frac{Hx\sqrt{x^2 - 1}}{x^2 - B^2}, \text{ so}$$

$$(Hx\sqrt{x^2 - 1})2x = H(x^2 - B^2)\left\{\sqrt{x^2 - 1} + \frac{x^2}{\sqrt{x^2 - 1}}\right\}$$

Multiply out

$$2x^2\sqrt{x^2 - 1} = \frac{(x^2 - B^2)(2x^2 - 1)}{\sqrt{x^2 - 1}}$$

Multiply further

$$2x^2(x^2 - 1) = (x^2 - B^2)\,(2x^2 - 1) \text{ or}$$

$$2x^4 - 2x^2 = (x^2 - B^2)\,(2x^2 - 1)$$

$$x^2 = \frac{B^2}{2B^2 - 1} \quad \therefore\ x = \frac{B}{\sqrt{2B^2 - 1}}$$

This value of x gives the turning point of the Q function, so it must be put in to find the value of the function, and equated to $\frac{1}{H}$.

$$\frac{\dfrac{B^2}{2B^2 - 1} - B^2}{\dfrac{HB}{\sqrt{2B^2 - 1}}\sqrt{\dfrac{1 - B^2}{2B^2 - 1}}} = H$$

This reduces to

$$\frac{2B^2 - 2B^4}{B\sqrt{1 - B^2}} = H^2$$

It further simplifies to

$$\frac{2\,(B - B^3)}{\sqrt{1 - B^2}} = H^2 \text{ or } \frac{2B\,(1 - B^2)}{\sqrt{1 - B^2}} = H^2.$$

$$2B\sqrt{1 - B^2} = H^2$$

Square this and it becomes

$$4B^2(1 - B^2) = H^4$$

$$4(B^2)^2 - 4(B^2) + H^4 = 0$$

This is a quadratic equation, and makes

$$B^2 = \frac{1 \pm \sqrt{1 - H^4}}{2}$$

This determines the function when H is chosen. Next find the unity value of the function for the peak frequency. This is

$$x^2 - \frac{1 \pm \sqrt{1 - H^4}}{2} = Hx\sqrt{x^2 - 1}$$

or

$$x^4(1 - H^2) - x^2(H^2 - 2B^2) + B^4 = 0.$$

The solutions are the peak frequencies which are sufficient to make the two sections.

The next thing if it is desired to calculate the lowest useful frequency is to equate the Q function, not to H this time, but to $\frac{1}{H}$, for the Q function $Q(2)$ reaches $\frac{1}{H}$ in its first fall from infinity.

Once H and the useful frequency range are decided, these equations give the frequencies of infinite attenuation. In making the filter, if the frequencies of infinite attenuation are known, one for each section, it is easy to make up the section.

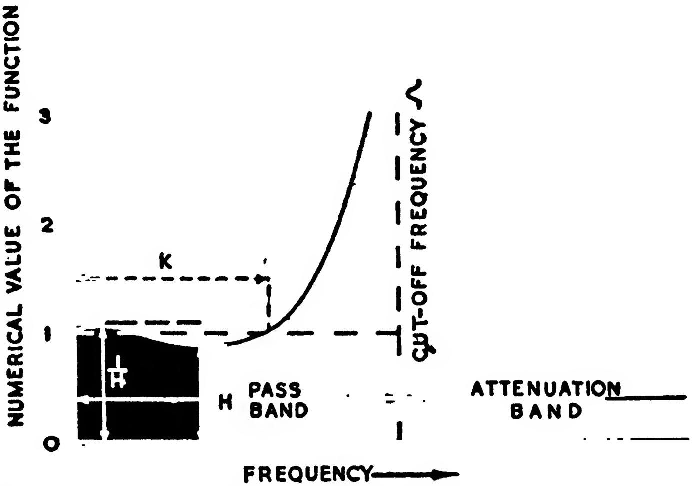


FIG. 99. THE IMPEDANCE VARIATION OF THE CAUER Q FUNCTION FOR IMPEDANCE IN THE PASS BAND

Ladder sections may be used rather than lattice sections. In either case the cut-off frequency, the characteristic impedance, and the frequency of infinite attenuation give the section. The calculation of the best peak frequencies for more than two sections is harder, and has been done by using elliptic functions.

Impedance Characteristics

Cauer not only used these Q functions for getting an attenuation curve which does not fall below a certain value, but he used the Q functions for $\sqrt{ab}$ in a lattice which is Z_0, the characteristic impedance. Then H is the greatest amount by which the impedance departs from its nominal value. There is again a useful range, so if $H = 0{\cdot}9$ the impedance if nominally 600 ohms varies between H times 600 ohms, which is 540 ohms, and $\frac{1}{H}$ times 600 ohms. This will be

up to a certain frequency just below cut-off. There is another "no man's land" here, where the impedance is too bad to be useful, going to 0 or infinity at the cut-off.

Fig. 99 shows a Q function used as an impedance frequency curve and how it is placed with regard to the value 1 used as the nominal impedance of the filter.

Here, again, Q functions can be used upside down, for it is always possible to make a filter having the reciprocal impedance of any other. In ladder filter design the termination settles whether it goes up to infinity or down to zero at the cut-off. The first three Q functions are shown below.

Writing $\frac{1}{\Omega}$ instead of Ω for any Q function gives the function for impedance.

The Fundamental Functions

$$Q(1) = \frac{\Omega_A}{H\sqrt{\Omega_A^2 - 1}}$$

$$Q(2) = \frac{H\Omega_A\sqrt{\Omega_A^2 - 1}}{\Omega_A^2 - B_1^2}$$

$$Q(3) = \frac{\Omega_A(\Omega_A^2 - B_2^2)}{H(\Omega_A^2 - B_1^2)\sqrt{\Omega_A^2 - 1}}$$

B_1 and B_2 are coefficients which finally settle the sizes of the coils and condensers. Their values vary according to the desired characteristics, even for the same cut-off frequency or frequencies.

The original Cauer design took one Q function to be used for attenuation and the H in it settled the desired attenuation as shown in the following table for $Q(1)$.

TABLE 21

SINGLE SECTION OF DERIVED LADDER FILTER

(Corresponds to Cauer's Class 1 or $Q(1)$)

db	Lowest Useful Frequency	Frequency of Infinite Attenuation
38	3·24	4·52
33	2·46	3·4
29	2	2·73
25·5	1·76	2·28
22·5	1·49	1·97
20	1·34	1·74
17·5	1·23	1·55
15·3	1·15	1·41
13·1	1·09	1·3
11	1·05	1·2

Another Q function was taken for impedance; and the H value in it is determined by the desired closeness to 1 of the impedance curve in the pass band.

When used in the pass band for impedance the table is as follows—

TABLE 22

PROPERTIES OF $Q(1)$
DATA FOR A CLASS I FILTER

db	Népers	k	γ	H
38·2	4·39	0·3090	0·2213	1·025
36·1	4·15	0·3420	0·2455	1·032
34·6	3·98	0·3746	0·2699	1·038
33	3·79	0·4067	0·2940	1·046
31·5	3·62	0·4384	0·3180	1·055
30·2	3·47	0·4695	0·3423	1·064

In the higher classes there is more than one peak attenuation; the third Q function goes through unity three times and so has three peaks of attenuation. To know the function completely one needs to decide what H is wanted and then what values of the coefficients B_1, B_2, etc., will give the troughs all on the same level in the attenuation curve. Having completely settled a function for attenuation $\sqrt{\frac{a}{b}}$ and one for impedance $\sqrt{ab}$, these can be multiplied and divided to determine the arms a and b since

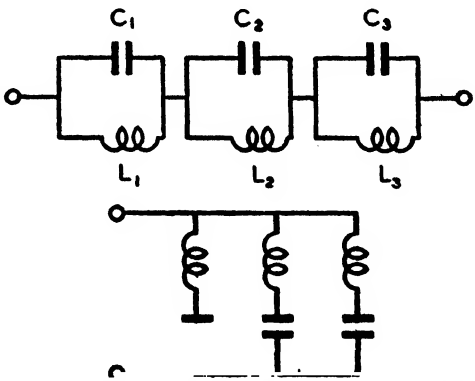
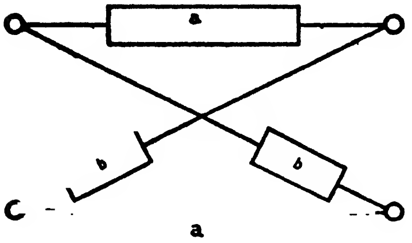


FIG. 100. (*Top*) AN ARM OF A LATTICE AS DEVELOPED BY CAUER
(*Bottom*) AN ALTERNATIVE ARRANGEMENT

$$\sqrt{\frac{a}{b}} \times \sqrt{ab} = a \text{ and } \sqrt{ab} \times \sqrt{\frac{b}{a}} = b$$

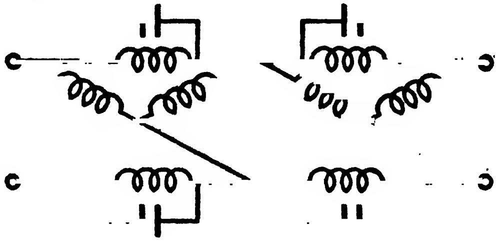
The arm of the original lattice is shown in Fig. 100.

We have shown how the peak of attenuation for a Class 1 filter depending on the $Q(1)$ function reaching 1 may be placed in the right position in the frequency scale to secure any desired result within the capabilities of a single ladder section to which the $Q(1)$ function belongs. We have also shown how the peaks are calculated for the $Q(2)$ function.



(a)

This matter of putting the attenuation peaks at properly chosen frequencies is a feature of first importance. It is easy to



(b)

FIG. 101. ONE COMPLICATED LATTICE SECTION EQUIVALENT TO TWO SIMPLE LATTICES

show that if two ladder-derived sections are placed end to end, each having a $Q(1)$ function for attenuation, the function $Q(2)$ expresses the attenuation of the two sections. This is useful, because it means that the study of the function $Q(2)$ gives information not only for lattice but for ladder filters too.

The work of the Russian mathematician Tschebbytcheff is used. The functions are known as Tschebbytcheff's polynomials. Fig. 101 (*a*) and (*b*) shows that a complicated lattice has an equivalent in the form of two simple lattices, and Fig. 102 shows a ladder filter having two sections, each of which corresponds to a lattice section in Fig. 101. Thus a ladder filter with derived sections corresponds to the complicated lattice originally studied by Cauer, as regards attenuation.

Design of Band Pass Filters

When it is a case of a low pass filter the symbol Ω_A is used to indicate $\dfrac{\text{frequency in use}}{\text{cut-off frequency}}$. Happily for high pass and low pass

and also band pass filters, the same curves or tables suffice. The Ω_A has a different meaning however. For a high pass filter it is $\dfrac{\text{cut-off frequency}}{\text{frequency in use}}$, and for a band pass filter, where the band is narrow compared with the frequency in the middle of the band, then the value of Ω_A becomes: $\dfrac{\text{frequency minus mid-band frequency}}{\text{half width of band in cycles}}$. Suppose we have a filter for 40–44 kc, and that it is desired to calculate

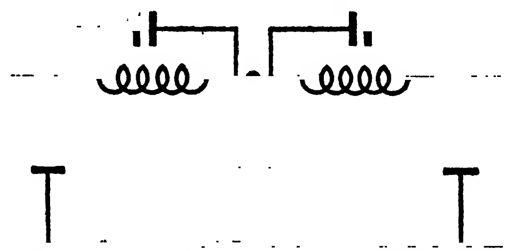


FIG. 102. THE TWO-SECTION LADDER EQUIVALENT TO TWO LATTICE SECTIONS

its attenuation at 46 kc. Then, because the mid-band frequency is 42 kc, it follows that $\Omega_A = \dfrac{46-42}{44-42} = 2$. Hence $\Omega_A = 2$ corresponds to 46 kc in this case. This makes the design of narrow band filters quite easy; for having got from curves or tables the Ω_A values of infinite attenuation, this formula gives the *actual* frequency of infinite attenuation and that enables the filter components to be calculated.

The peak frequencies are given by Cauer's methods in the following form. He has, as already mentioned, several classes, and each has peaks of attenuation whose places depend on how near to the cut-off one must work. In other words, one knows the lowest frequency one wants to use, called the "lowest usable frequency," and the tables give, for each class, the attenuation at and beyond that frequency and also the peaks of attenuation at and beyond the "lowest useful frequency" (see Table 21). A 10 kc low pass filter, if desired to have 20 db attenuation, will give it above 13·4 kc. The peak must then be made at 17·4 kc.

CHAPTER XI

FILTERS USING CIRCUITS WITH UNUSUAL ELEMENTS

Quartz Crystal Filters

LATELY it has been found that a flat piece of quartz placed between two metal plates gives a "reactive" voltage when a pressure is fed to the electrodes. This is due to the so-called "Piezo effect."

Sometimes—as a rule in fact—the faces of the quartz are covered with metal by sputtering, as it is called, i.e. striking an arc with the gold or aluminium which it is desired to put on the crystal. This forms the electrodes. A voltage applied to the electrodes causes the crystal to alter in shape. Sometimes it is an expansion or contraction in a direction parallel to the electrodes, sometimes in a direction at right angles to the electrodes. This is the Piezo effect, or part of it. Alteration in the shape of the quartz develops an electric charge on the electrodes, and if an alternating e.m.f. is applied to the crystal, alternating charges appear on the electrodes to which the voltage is applied. Since a changing charge is a current, i.e. one coulomb per second equals one ampere, currents flow in the leads to the crystal just as if it were a coil, a condenser, or a resistance. Indeed, it behaves like all three in turn when the frequency is altered. This is because the applied voltage that bends it and the currents produced by its mechanical motion when bent depend, as regards phase, on the mechanical phase relation between force and motion, i.e. velocity of the crystal.

How does a vibrating plate move as regards phase of motion when acted on by force in mechanics? The answer is that everything depends on frequency, whether the frequency of the alternating force is below, near to, or above the frequency of resonance (mechanical resonance, that is) of the crystal. Below resonance the stiffness counts; much above it the mass; at resonance the frictional losses control the motion. Further, at mechanical resonance the movement is large, and so are the electric charges due to Piezo action. The current, therefore, is quite large and may be 10 milliamps. for one volt applied to the crystal, which, therefore, lookslike a mere 100 ohms in that case. Above resonance, as we have said, the mass controls the movement in accordance with Newton's Laws of Motion.

Acceleration is proportional to E where E is the applied electric voltage which bends the crystal.

Since, however, acceleration is $\frac{dv}{dt}$ where v means velocity, and velocity is the cause of current (because it is change of or rate of change of position and position means a charge) it follows that because

$$Ma \propto E \text{ and } I \propto v$$

where a is acceleration, then

$$I \propto \frac{dE}{dt}$$

Thus there is a 90° phase lag between current and voltage at high frequency. As the frequency is raised the effect of the mass is to

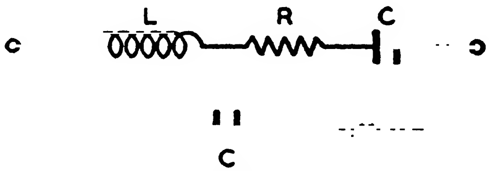


Fig. 103

make the crystal stand still, hence there is no current at an infinite frequency.

When the correct sign is put into $I \propto \frac{dE}{dt}$ it is found that $E = j$ constant $(\omega)I$, which may be written $E = jL\omega I$, so the crystal appears to have a high *inductance*. Much below resonance it is a leading current just like a condenser, so the whole circuit acts like a condenser, a resistance, and an inductance in series. In addition, the electrodes with the quartz between act like a small condenser, a much larger condenser, however, than the equivalent capacity due to the Piezo action. Thus the crystal circuit is as shown in Fig. 103.

The crystal acts like a low impedance near resonance, but owing to the electrode capacity there is an anti-resonance very near to the resonance point—just a few hundred cycles away, maybe. It is at a slightly higher frequency than the mechanical resonant frequency because the whole can only come into anti-resonance when the three

series elements look like an inductance of reactance comparable and nearly equal to that of the electrode capacity.

The electrode capacity rather complicates the action of the crystal, and often has to be balanced out. One way to do this is to use a tapped coil such as the Heaviside equal ratio bridge. A good "line pass filter" to pass a carrier only, i.e. a very sharp resonance, may be made, thus (Fig. 104)—

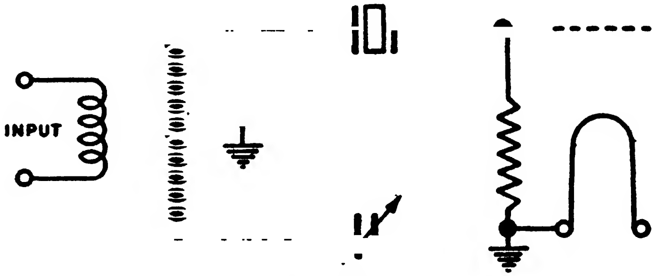


FIG. 104

The small variable condenser is used to balance the electrode capacity of the crystal. The band width is a cycle or two. For 200–300 cycles band width two crystals may be used of different natural frequency, any difference in their electrode capacities being made up by a variable condenser in parallel with the smaller, thus (Fig. 105)—

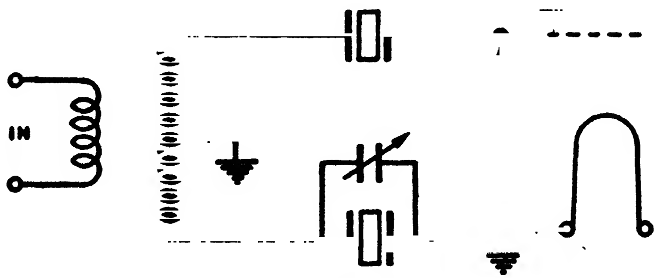


FIG. 105

The condenser may be a differential one.

It is possible to make a gate type filter with additional coils in parallel or else in series, thus (Fig. 106)—

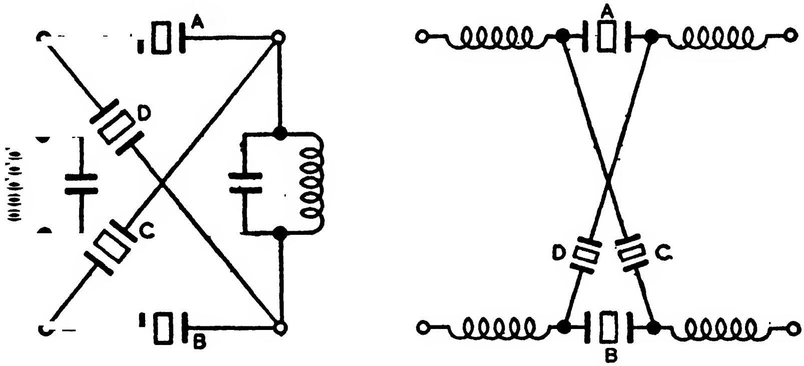


FIG. 106

In both cases the crystals *A* and *B* should be of the same *L* and *C* value. So should *C* and *D*. One pair, i.e. *A* and *B* or else *C* and *D*, should have additional condensers so that all four electrode capacities can be made correct.

Co-axial Cable Filters

If pieces of co-axial cable are "T"'d into the co-axial cable at regular intervals, the inner and outer conductors having been correctly connected at the junctions, a filter results. (See Fig. 107.)

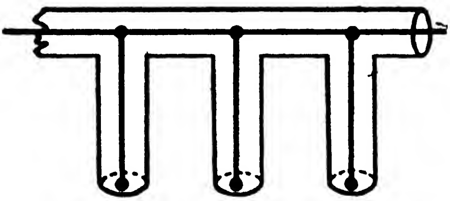


FIG. 107. A CO-AXIAL CABLE FILTER

The ends of the "T" pieces may be short-circuited, which is the condition the sketch is meant to represent.

Condensers can be included in the co-axial line by breaking the inner conductor and using a ceramic disc inside the outer conductor, thus (Fig. 108).

Similarly, a shunt condenser may be put across the inner and outer conductors of the cable. This requires a ceramic cylinder with

the outer tube forming one conductor plate and a cylindrical electrode inside it connected to the inner cable conductor forming the other plate.

The calculation of the pass and attenuation bands of such filters is not very easy because the simple coil $jL\omega$ or condenser $\frac{1}{jC\omega}$ in a ladder or lattice circuit is replaced by a piece of cable having an

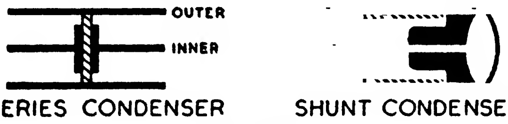


Fig. 108

impedance, because it is short-circuited at the far end of $Z_0 \tanh Pl$. If it is open-circuited it is the coth. Table 23 shows a few co-axial cable filters.

The sketch (Fig. 109) shows the physical construction of a typical co-axial cable filter.

Wave Guide Filters

The invention of oscillators for very high frequency work, namely, over 1000 megacycles, has made possible the use of hollow tubes of

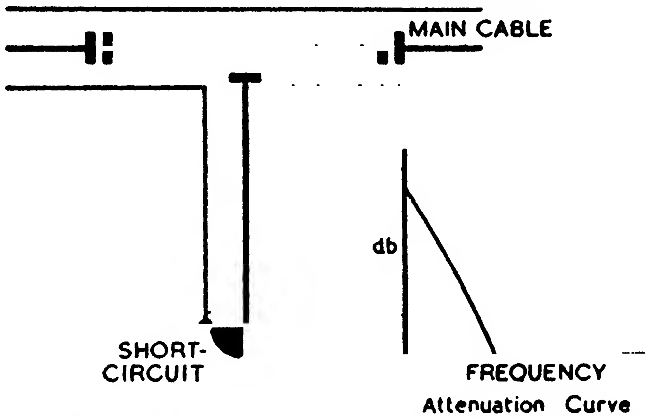
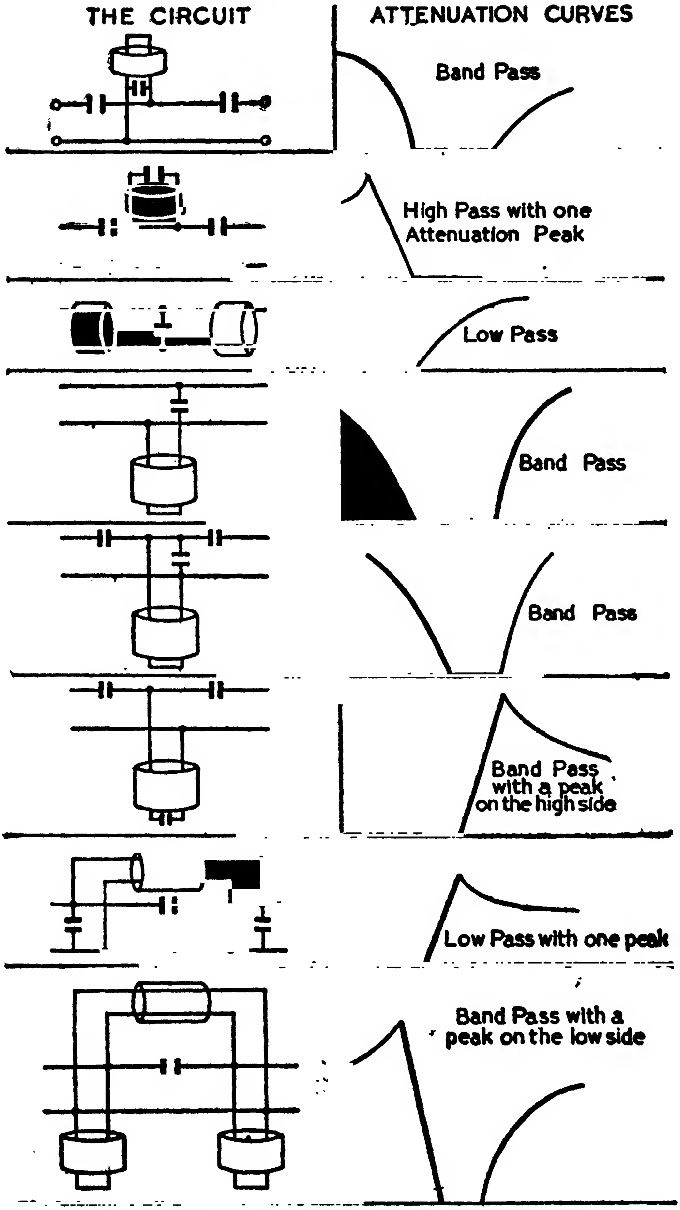


Fig. 109. Sketch showing the Physical Construction of a Typical Co-axial Cable Filter

reasonable size as "wave guides." There are two types of waves, one called "electric" or "E" type and another called "magnetic" or "H" type waves.

TABLE 23

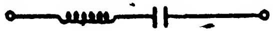
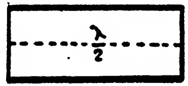
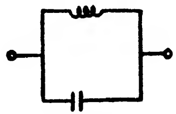
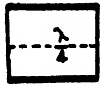
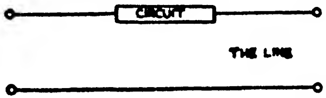
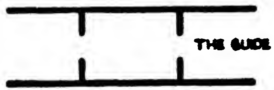
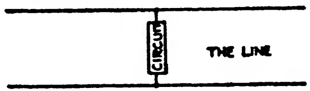
CO-AXIAL CABLE FILTERS *



* Courtesy of the *Bell System Technical Journal.*

Table 24

The Analogy between Wave Guide Resonators and Ordinary Tuned Circuits

FEATURE	ORDINARY CIRCUITS	FEATURE	WAVE GUIDE WORK
THE SERIES TUNED CIRCUIT		HALF WAVE RESONATOR	
PARALLEL TUNED CIRCUIT		QUARTER WAVE RESONATOR	
A CIRCUIT PLACED "IN" THE LINE		A RESONATOR "IN" THE GUIDE	
A CIRCUIT ACROSS THE LINE		A RESONATOR T'EED INTO THE LINE	
THE Q VALUE	$\frac{LW}{R}$	THE Q VALUE	SETTLED BY THE METAL USED AND SIZE, FOR A GIVEN SHAPE
DYNAMIC IMPEDANCE	$\frac{L}{CR}$ FOR PARALLEL CIRCUIT	THE "IMPEDANCE"	THE SIZE OF ORIFICE OR ORIFICES, IN RELATION TO CAVITY SIZE.

Both contain electric and magnetic forces and fluxes, but there is no magnetic flux along the pipe in an electric or "E" wave—it is all in circles—and there is no electric force or displacement current in an axial direction in the magnetic wave—here the electric force is in circles round the pipe.

These waves can only exist without very rapid attenuation if the pipe diameter is larger than a certain critical value depending on the type and "order" of the wave. Thus a narrow pipe is a high pass filter.

A curious and remarkable filter can be constructed for "E" waves by placing a number of copper "spokes" in the tube. Since the electric force is radial these short-circuit it, but allow an "H" wave to pass.

There are three other forms of filter for wave guides which may be possible. These are—

(1) Two pipes differing by an odd number of odd half wavelengths for the frequency being used which it may be desirable to attenuate. By the word "wavelength" is here meant wavelength in the tube, which is not the same as in an unbounded medium such as air.

(2) Closed resonators with a small tube between to "couple" them.

(3) A wave guide with pieces of tube put in as "T" pieces on the analogy of the co-axial and acoustic filters which are constructed of "T" pieces put into a tube at intervals. The analogy between the co-axial and the present wave guide filter is not a perfect one, but there is no doubt that curious and probably very useful results will be obtained by using guides and resonators of specific geometrical shape.

CHAPTER XII

CONCLUSION

In view of the author's Filter Theorem that any set of reactances (coils and condensers) will make a filter, however they may be connected, it is felt that every circuit possible ought to be mathematically investigated before one can say that one has found the best circuit for a given number of coils and condensers.

Cauer's work, great as it is, does not appear to the writer to spell finality, for his method applies fundamentally to lattice networks. The Cauer lattice may be complicated and one may even be able to design a ladder equivalent say, but there may still be another circuit connection for which there may be no ladder or lattice equivalent (because negative capacity may be wanted).

Fig. 93 shows a lattice, but Fig. 101 shows something more complete. Why should not this be a better filter than anything one can make out of a lattice with the same number of coils and condensers?

The writer has found when doing research work on filters at Liverpool University that one can design and build a filter of very sharp cut-off and good attenuation with only two coils, say, using neither a lattice nor a ladder, there being no simple ladder equivalent of the circuit in use.

The method, then, would seem to be, with each and every number of coils and condensers, to draw every circuit which it is possible to draw, and then to examine each for "open" and "closed" impedance and so form Q functions which will be *modified* Q functions for Z_0 and tanh $\frac{P}{2}$, modified because the circuit may bring in, say, $(x^2 - B^2)^2$ and so the Tschebbytcheff polynomials as used by Cauer might not be suitable.

The point is that $Q(n)$ needs so many coils and condensers all of which must be independent and so capable of being made a different size if the Q function calls for it.

A certain circuit tried may, however, be so complicated as to give a complicated Q function, with, say, two B's the same, i.e.

$$\frac{x}{H}\,\frac{\sqrt{(x^2-1)}\,(x^2-B_2^2)}{(x^2-B_1^2)\,(x^2-B_1^2)}$$

It may be objected that this is not as fluxible a function as the ordinary Q function, where no two of the factors are the same. That is true, but it may yet be far better than the simple Q function which would arise from the small number of coils and condensers which produced this special function, and the result may be a gain in attenuation and impedance.

APPENDIX I

DESIGN OF FILTERS FOR OPERATION OVER A RESTRICTED FREQUENCY RANGE

SUPPOSE a carrier system operates on a total frequency band which has an upper limit of 50 kc.

When the band pass filters are designed, it may be that a simpler filter circuit can be made, if the attenuation is allowed to fall away above 50 kc.

In other words, instead of writing the requirements of a filter thus: "The attenuation is to be low between 26 kc and 30 kc, but above 70 db below 25 kc and above 31 kc," the requirements are modified as follows—

"The attenuation is to be low between 26 kc and 30 kc, but, though it must be over 70 db from 31 kc to 50 kc, it may fall again after 50 kc." It will exceed 70 db in the region below 25 kc, but if transformed thus, will have a smaller attenuation down at the low-frequency end.

When a filter is specified thus over a restricted frequency range ONLY, there is difficulty in calculating where the peak attenuations ought to be placed to secure the required performance. Cauer's method is to use what he calls a "frequency transformation." Since he uses one variable Ω for frequency in all his curves (and tables), what is needed is to be able to say what the Ω value is in the table, corresponding to any and every frequency in the "spectrum." For a low pass filter without any frequency transformation, this is easy. If the cut-off is 10 kc, and in plotting a graph, or for design information, we desire the characteristics at 12 kc, then $\Omega = 1{\cdot}2$, because this general Ω is $\frac{12}{10}$, being

$$\frac{\text{Frequency}}{\text{Cut-off frequency}}$$

in the case of a low pass filter.

When it comes to a high pass filter,

$$\Omega = \frac{\text{Cut-off frequency}}{\text{Frequency}}$$

and for a band pass filter it is again another function of frequency.

Thus the relation between the Ω values in Cauer's tables and the actual frequency as marked on the oscillator varies with the type of filter. THIS RELATION IS MORE COMPLICATED WHEN FILTERS ARE MADE WITH A RESTRICTED FREQUENCY RANGE.

So that is what Cauer does. He makes all the peaks of attenuation fall within the desired range and, to place them in their best positions, formulas connecting the frequency and his "universal" Ω are needed. He gives these formulas.

APPENDIX II

THE EFFECT ON THE PERFORMANCE OF A FILTER OF SLIGHT LACK OF ACCURACY IN COILS AND CONDENSERS

EVERYTHING depends on two factors. These are: (1) the purpose for which a filter is designed, and (2) whether it is a wide or narrow band filter.

In the design of a carrier system, it is desirable to have as much margin as possible between, for example, the cut-off frequency of the line low pass filter and the upper end of the range of telephone frequencies, so that if the filter alters in its cut-off due to inaccuracy in the coils and condensers, no harm will result in the transmission.

The following rules are useful—

Rules for Determining the Effect of Inaccuracy in Filters

(1) A 1 per cent increase in every component of a filter gives it the same impedance, attenuation and insertion loss curves if on the graphs all frequency figures are made 1 per cent LOW.

(2) An increase of 1 per cent in all condensers only gives a lowering of the frequency figures of $\frac{1}{2}$ per cent.

(3) Provided all the tuned circuits have their correct resonant frequencies, the mere change of a coil to a higher value and condenser to a lower value cannot alter the total impedance of the filter much, except at places where it is nearly 0 or infinity, which is near the cut-off.

(4) A general increase in coil inductance of 5 per cent and decrease of capacity of 5 per cent may, or rather will, alter the impedance at every frequency by + 5 per cent.

(5) Such an increase or decrease of impedance cannot alter the insertion loss by more than a fraction of 1 db.

(6) It appears that lack of accuracy in the components of coils may be due to self-capacity.

The self-capacity of certain coils is rather high. The remedy, if this is important, is to add a spacer to each porcelain piece, giving four rather than two sections of coil. Inaccuracy in most of the sections does not matter as much as in a particular section.

(7) The section to be watched is that *m* section which is nearest in its peak frequency to the cut-off frequency. Errors in the components of that particular *m* section have the greatest influence on the insertion loss curve.

(8) Band pass filters with high mid-band frequency are more critical than band filters with the same width of band but lower mid-band frequency.

INDEX

ATTENUATION, 1, 34
—— and Cosh P, 34
—— equalizer chart, 113
—— equalizers, 107
—— —— (constant resistance), 111
—— measurement, 124
Attenuator circuits, 8
—— (values for "T"), 7
Attenuators, 1

BALANCE to earth, 11
Band pass filter, 41
—— —— —— (components), 44
—— —— ——, simple explanation of, 47
—— —— —— terminations, 46
Bridged "T" networks, 91
Bridges, 114

CABLE, coil-loaded, 12
Campbell's theorem, 14, 130
Cauer filters, 139
Characteristic impedance of ladders, 13
—— —— of lattices, 66
Coil winding (resistances), 9
Composite filters, 64
Cosh x, 29

DERIVED band pass filter, 59
—— high pass filter, 59
—— ladder filter impedances, 57
—— —— filters, 50
—— low pass filter, 50

EQUIVALENT ladder to lattice, 69
—— networks, 88

FILTER theorem, author's, 130
Formula for attenuation of a ladder, 23
—— —— characteristic impedance of a ladder, 13
—— —— components of ladder filters, 49
—— —— impedances of a line open and closed, 25
Foster's theorem, 132
Frequency bridge, 120

GATE networks, 89

HALF-SERIES termination, 12, 13
High pass filter, 36
Hybrid coil bridge, 121

IMPEDANCE multiplier theorem, 131
—— of ladder low pass filter (double-shunt), 21, 22
—— —— —— —— —— (half-series), 17
Insertion loss chart, 83
—— —— curves, 85, 86
—— losses, 79
Inversion of impedances, 134
—— of low pass into high pass filter, 37

JOINING sections together, 54

LADDER band pass filter, simple explanation of, 47
—— low pass filter, simple explanation of, 30
"L" attenuation equalizer, 108
Lattice band pass filter, 70
—— filters, 66
—— ——, criterion for a pass band, 72
—— low pass filter, 68
Linearity theorem, 137
Losses, general remarks on, 77
—— in components, 73

MAXIMUM volt-ampere theorem, 128
Measurements, 114
Mutual inductance, how to deal with, 94

NEGATIVE series bridge, 116
—— shunt bridge, 118

OPEN and closed cable formulas, 25

PHASE changes in filters, 31, 35
—— measurement, 123
—— shift networks, 100
Positive series bridge, 120
—— shunt bridge, 119

"Q" FUNCTIONS, 146
Quartz crystal filters, 158

Reciprocal theorem, 137
Resistance manipulation of formulas of circuits, 77
—— winding, 12

Simplicity of alternating currents, 139
Star mesh theorem, 92

Tanh x, 68
Terminating filters, 13

Theorems, 126
Thévénin's theorem, 127
Trigonometry, use of in avoiding arithmetic, 148
Tuned transformer, band pass filter, 95

Vectors at nearly 90°, 74

Wave guide filters, 164